A Generalization of Riemann Sums

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Abstract

We generalize the property that Riemann sums of a continuous function corresponding to equidistant subdivisions of an interval converge to the integral of that function. We then give some applications of this generalization.

Problem U131 in [1] reads:

Prove that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\arctan\frac{k}{n}}{n+k}\cdot\frac{\varphi(k)}{k}=\frac{3\log 2}{4\pi}, \tag{1}$$

where φ denotes Euler's totient function. In this note we prove the following theorem, that will, in particular, answer this question.

Theorem 1. *Let α be a positive real number and let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n}a_k=L.$$

For every continuous function f on the interval $[0,1]$,

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)a_k=L\int_0^1\alpha x^{\alpha-1}f(x)\,dx.$$

Proof. We use the following two facts:

fact 1 for $\beta>0$

$$\lim_{n\to\infty}\frac{1}{n^{\beta+1}}\sum_{k=1}^{n}k^\beta=\frac{1}{\beta+1}$$

fact 2 if $(\lambda_n)_{n\geq 1}$ is a real sequence that converges to 0, and $\beta>0$ then

$$\lim_{n\to\infty}\frac{1}{n^{\beta+1}}\sum_{k=1}^{n}k^\beta\lambda_k=0.$$

Indeed, *fact 1* is just the statement that the Riemann sums of the function $x\mapsto x^\beta$ corresponding to an equidistant subdivision of the interval $[0,1]$ converges to $\int_0^1 x^\beta\,dx$.

The proof of *fact 2* is a "Cesáro" argument. Since $(\lambda_n)_{n\geq 1}$ converges to 0 it must be bounded, and if we define $\Lambda_n = \sup_{k\geq n} |\lambda_k|$, then $\lim_{n\to\infty} \Lambda_n = 0$. But, for $1 < m < n$, we have

$$\begin{aligned}\left|\frac{1}{n^{\beta+1}}\sum_{k=1}^{n} k^{\beta}\lambda_k\right| &\leq \frac{1}{n^{\beta+1}}\sum_{k=1}^{m} k^{\beta}|\lambda_k| + \frac{1}{n^{\beta+1}}\sum_{k=m+1}^{n} k^{\beta}|\lambda_k| \\ &\leq \frac{m^{\beta+1}}{n^{\beta+1}}\Lambda_1 + \Lambda_m.\end{aligned}$$

Let ϵ be an arbitrary positive number. There is an $m_\epsilon > 0$ such that $\Lambda_{m_\epsilon} < \epsilon/2$. Then we can find $n_\epsilon > m_\epsilon$ such that for every $n > n_\epsilon$ we have $m_\epsilon^{\beta+1}\Lambda_1/n^{\beta+1} < \epsilon/2$. Thus

$$n > n_\epsilon \quad \Longrightarrow \quad \left|\frac{1}{n^{\beta+1}}\sum_{k=1}^{n} k^{\beta}\lambda_k\right| < \epsilon.$$

This ends the proof of *fact 2.*

Now, we come to the proof of our Theorem. We start by proving the following property by induction on p :

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+p}}\sum_{k=1}^{n} k^p a_k = \frac{\alpha}{\alpha+p}L. \tag{2}$$

The base property ($p = 0$) is just the hypothesis. Let us assume that this is true for a given p and let

$$\lambda_n = \frac{1}{n^{\alpha+p}}\sum_{k=1}^{n} k^p a_k - \frac{\alpha L}{\alpha+p},$$

(with the convention $\lambda_0 = 0$,) so that $\lim_{n\to\infty} \lambda_n = 0$. Clearly,

$$k^p a_k = k^{\alpha+p}\lambda_k - (k-1)^{\alpha+p}\lambda_{k-1} + \frac{\alpha L}{\alpha+p}\left(k^{\alpha+p} - (k-1)^{\alpha+p}\right),$$

hence

$$\begin{aligned}k^{p+1}a_k &= k^{\alpha+p+1}\lambda_k - k(k-1)^{\alpha+p}\lambda_{k-1} + \frac{\alpha L}{\alpha+p}\left(k^{\alpha+p+1} - k(k-1)^{\alpha+p}\right), \\ &= k^{\alpha+p+1}\lambda_k - (k-1)^{\alpha+p+1}\lambda_{k-1} + \frac{\alpha L}{\alpha+p}\left(k^{\alpha+p+1} - (k-1)^{\alpha+p+1}\right) \\ &\qquad - (k-1)^{\alpha+p}\lambda_{k-1} - \frac{\alpha L}{\alpha+p}(k-1)^{\alpha+p}\end{aligned}$$

It follows that

$$\frac{1}{n^{\alpha+p+1}}\sum_{k=1}^{n} k^{p+1}a_k = \lambda_n - \frac{1}{n^{\alpha+p+1}}\sum_{k=1}^{n-1} k^{\alpha+p}\lambda_k + \frac{\alpha L}{\alpha+p}\left(1 - \frac{1}{n^{\alpha+p+1}}\sum_{k=1}^{n-1} k^{\alpha+p}\right).$$

Using *fact 1* and *fact 2* we conclude that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+p+1}}\sum_{k=1}^{n}k^{p+1}a_k=\frac{\alpha L}{\alpha+p}\left(1-\frac{1}{\alpha+p+1}\right)=\frac{\alpha L}{\alpha+p+1}.$$

This ends the proof of (2).

For a continuous function f on the interval $[0,1]$ we define

$$I_n(f)=\frac{1}{n^{\alpha}}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)a_k,\quad\text{and}\quad J(f)=L\int_0^1\alpha x^{\alpha-1}f(x)\,dx.$$

Now, if X^p denotes the function $t\mapsto t^p$, then (2) is equivalent to the fact that $\lim_{n\to\infty}I_n(X^p)=J(X^p)$, for every nonnegative integer p. Using linearity, we conclude that $\lim_{n\to\infty}I_n(P)=J(P)$ for every polynomial function P.

On the other hand, if $M=\sup_{n\geq 1}\frac{1}{n^{\alpha}}\sum_{k=1}^{n}a_k$, then $L\leq M$ and we observe that for every continuous functions f and g on $[0,1]$ and all positive integers n,

$$|I_n(f)-I_n(g)|\leq M\sup_{[0,1]}|f-g|\quad\text{and}\quad|J(f)-J(g)|\leq M\sup_{[0,1]}|f-g|\,.$$

Consider a continuous function f on $[0,1]$. Let ϵ be an arbitrary positive number. Using Weierstrass Theorem there is a polynomial P_ϵ such that $\|f-P_\epsilon\|_\infty=\sup_{x\in[0,1]}|f(x)-P_\epsilon(x)|<\frac{\epsilon}{3M}$. Moreover, since $\lim_{n\to\infty}I_n(P_\epsilon)=J(P_\epsilon)$, there exists an n_ϵ such that $|I_n(P_\epsilon)-J(P_\epsilon)|<\frac{\epsilon}{3}$ for every $n>n_\epsilon$. Therefore, for $n>n_\epsilon$, we have

$$|I_n(f)-J(f)|\leq|I_n(f)-I_n(P_\epsilon)|+|I_n(P_\epsilon)-J(P_\epsilon)|+|J(P_\epsilon)-J(f)|<\epsilon.$$

This ends the proof of Theorem 1. $\square$

Applications.

- It is known that Euler's totient function φ has very erratic behaviour, but on the mean we have the following beautiful result, see [2, 18.5],

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\varphi(k)=\frac{3}{\pi^2}.\tag{3}$$

  Using Theorem 1 we conclude that, for every continuous function f on $[0,1]$,

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\varphi(k)=\frac{6}{\pi^2}\int_0^1 xf(x)\,dx.\tag{4}$$

Choosing $f(x) = \frac{\arctan x}{x(1+x)}$ we conclude that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\arctan(k/n)}{k(n+k)}\varphi(k) = \frac{6}{\pi^2} \int_0^1 \frac{\arctan x}{1+x}\,dx. \tag{5}$$

Thus we only need to evaluate the integral $I = \int_0^1 \frac{\arctan x}{1+x}\,dx$. The "easy" way to do this is to make the change of variables $x \leftarrow \frac{1-t}{1+t}$ to obtain

$$\begin{aligned} I &= \int_0^1 \arctan\left(\frac{1-t}{1+t}\right)\frac{dt}{1+t} = \int_0^1 \left(\frac{\pi}{4} - \arctan t\right)\frac{dt}{1+t} \\ &= \frac{\pi}{4}\int_0^1 \frac{dt}{1+t} - I \end{aligned}$$

Hence, $I = \frac{\pi}{8}\log 2$. Replacing back in (5) we obtain (1).

- Similarly, if $\sigma(n)$ denotes the sum of divisors of n, then (see [2, 18.3]),

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} \sigma(k) = \frac{\pi^2}{12}.$$

Using Theorem 1 we conclude that, for every continuous function f on $[0, 1]$,

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\sigma(k) = \frac{\pi^2}{6}\int_0^1 xf(x)\,dx.$$

Choosing for instance $f(x) = \frac{1}{1+ax^2}$ we conclude that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sigma(k)}{n^2 + ak^2} = \frac{\pi^2}{12a}\log(1+a).$$

- Starting from

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\varphi(k)}{k} = \frac{6}{\pi^2},$$

which can be proved in the same way as (3), we conclude that, for every $\alpha \geq 0$,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+1}}\sum_{k=1}^{n} k^{\alpha-1}\varphi(k) = \frac{6}{\pi^2(1+\alpha)} \tag{6}$$

Also,

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^{\alpha-1}\log(k/n)\varphi(k)=\frac{6}{\pi^2}\int_0^1 x^\alpha\log(x)\,dx$$
$$=-\frac{6}{\pi^2(\alpha+1)^2}.$$

Hence, using (6), for $\alpha\geq 0$ we obtain:

$$\frac{1}{n^{\alpha+1}}\sum_{k=1}^{n}k^{\alpha-1}\log k\,\varphi(k)=\frac{6\big((1+\alpha)\log n-1\big)}{\pi^2(1+\alpha)^2}+o(1).$$

References

[1] C. Lupu, Problem U131, *Mathematical Reflections.* **(4)** (2009).

[2] G. H. Hardy AND E. M.Wright, An Introduction to the Theory of Numbers (5th ed.), *Oxford University Press.* (1980).

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