An alternate sum with zeta function

Proposed problem for MATHPROBLEMS

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The Problem : *Evaluate* $\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\zeta(2n)}{n}$, *where* ζ *is the Riemann's zeta function.*

Solution 1: We make use of the well known facts

$$\lim_{n\to+\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)}=\Gamma(z) \qquad z\in\mathbb{C}\setminus-\mathbb{N}_0, \tag{1}$$

$$\Gamma(z+1)=z\Gamma(z) \qquad z\in\mathbb{C}\setminus-\mathbb{N}_0, \tag{2}$$

$$\Gamma(1-z)\Gamma(z)=\frac{\pi}{\sin(\pi z)} \qquad z\in\mathbb{C}\setminus\mathbb{Z}_0 \tag{3}$$

We have:

$$\begin{aligned}\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\zeta(2n)}{n}&=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{+\infty}\frac{1}{k^{2n}}\\&=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\left(1+\sum_{k=2}^{+\infty}\frac{1}{k^{2n}}\right)\\&=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}+\sum_{n=1}^{+\infty}\sum_{k=2}^{+\infty}\frac{(-1)^{n-1}}{nk^{2n}}\\{}^{1}&=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}+\sum_{k=2}^{+\infty}\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\left(1/k^2\right)^n}{n}\\&=\ln 2+\sum_{k=2}^{+\infty}\ln\left(1+\frac{1}{k^2}\right)\\&=\sum_{k=1}^{+\infty}\ln\left(1+\frac{1}{k^2}\right).\end{aligned}$$

Now on account of (1),(2) and (3) we have

$$\begin{aligned}
\sum_{k=1}^{+\infty}\ln\left(1+\frac{1}{k^2}\right) &= \ln\left(\lim_{n\to+\infty}\prod_{k=1}^{n}\frac{k^2+1}{k^2}\right)\\
&= \ln\left(\lim_{n\to+\infty}\frac{\frac{1}{n!n^{1-i}}\prod_{k=0}^{n}(k+1-i)\frac{1}{n!n^{1+i}}\prod_{k=0}^{n}(k+1+i)}{\frac{1}{n!n}\prod_{k=0}^{n}(k+1)\frac{1}{n!n}\prod_{k=0}^{n}(k+1)}\right)\\
&= \ln\left(\frac{(\Gamma(1))^2}{\Gamma(1-i)\Gamma(1+i)}\right)\\
&= \ln\left(\frac{1}{\Gamma(1-i)i\Gamma(i)}\right)\\
&= \ln\left(\frac{\sin(\pi i)}{\pi i}\right)\\
&= \ln\left(\frac{\sinh\pi}{\pi}\right)
\end{aligned}$$

Solution 2: We make use of the well known fact (see [1] p.217)

$$\sum_{k=-\infty}^{\infty}\frac{1}{z+k}=\frac{\pi}{\tan(\pi z)}. \tag{4}$$

For $z\in\mathbb{C}$ with $|z|<1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\zeta(2n)z^{2n} &= \sum_{n=1}^{\infty}\left(z^{2n}\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\left(\frac{z^2}{k^2}\right)^n\right)\\
&= \sum_{k=1}^{\infty}\frac{z^2}{k^2}\cdot\frac{1}{1-\frac{z^2}{k^2}}=z^2\sum_{k=1}^{\infty}\frac{1}{k^2-z^2}\\
&= \frac{z}{2}\sum_{k=1}^{\infty}\left(\frac{1}{k-z}-\frac{1}{k+z}\right)\\
&= -\frac{z}{2}\sum_{k=1}^{\infty}\left(\frac{1}{z-k}+\frac{1}{z+k}\right)\\
&= -\frac{z}{2}\left(-\frac{1}{z}+\sum_{k=-\infty}^{\infty}\frac{1}{z+k}\right)\\
&= \frac{1}{2}-\frac{z}{2}\sum_{k=-\infty}^{\infty}\frac{1}{z+k}
\end{aligned}$$

[1] The double sum converges absolutely, compared with $\sum_{k\geq 2}\sum_{n\geq 1}\frac{1}{k^{2n}}$.

so on account of (4) and since $\lim\limits_{z\to 0}\frac{1}{2z}-\frac{\pi}{2\cdot\tan(\pi z)}=0$ we have

$$\sum_{n=1}^{\infty}\zeta(2n)\,z^{2n-1}=\frac{1}{2z}-\frac{\pi}{2\cdot\tan(\pi z)},\qquad |z|<1.$$

Now integrating, for $|w|<1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\zeta(2n)\int_0^w z^{2n-1}dz &= \int_0^w\left(\frac{1}{2z}-\frac{\pi}{2\cdot\tan(\pi z)}\right)dz \Rightarrow\\
\sum_{n=1}^{\infty}\frac{\zeta(2n)\,w^{2n}}{2n} &= \frac{1}{2}\left(\ln(z)-\ln(\sin(\pi\cdot z))\right)\Big|_0^w\\
&= \frac{1}{2}\ln\left(\frac{w}{\sin(\pi\cdot w)}\right)-\frac{1}{2}\lim_{z\to 0}\ln\left(\frac{z}{\sin(\pi\cdot z)}\right)\\
&= \frac{1}{2}\ln\left(\frac{w}{\sin(\pi\cdot w)}\right)+\frac{1}{2}\ln(\pi)\Rightarrow\\
\sum_{n=1}^{\infty}\frac{\zeta(2n)\,w^{2n}}{n} &= \ln\left(\frac{w}{\sin(\pi\cdot w)}\right)+\ln(\pi).
\end{aligned}$$

But since $\sum\limits_{n=1}^{+\infty}(-1)^n\frac{\zeta(2n)}{n}=\sum\limits_{n=1}^{+\infty}\frac{(-1)^n}{n}+\sum\limits_{n=1}^{+\infty}\sum\limits_{k=2}^{+\infty}\frac{(-1)^n}{nk^{2n}}$ and the two sums converge, (the second being absolutely convergent), $\sum\limits_{n=1}^{+\infty}(-1)^n\frac{\zeta(2n)}{n}$ converges and from Abel's theorem

$$\begin{aligned}
\sum_{n=1}^{+\infty}(-1)^n\frac{\zeta(2n)}{n} &= \lim_{w\to i}\ln\left(\frac{w}{\sin(\pi\cdot w)}\right)+\ln(\pi)\\
&= \ln\left(\frac{\pi}{\sinh\pi}\right).
\end{aligned}$$

The result is immediate.

References

[1] T.J.I'A. Bromwich *An Introduction to the Theory of Infinite Series*, MacMillan and Co, 1942

Evaluation of an infinite sum

Proposed problem for MATHPROBLEMS

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The Problem : *Evaluate the sum* $\displaystyle\sum_{n=1}^{+\infty} n\left(2^{-1/2}-1+\binom{1/2}{1}\frac{1}{2}-\binom{1/2}{2}\frac{1}{4}+\cdots+(-1)^{n+1}\binom{1/2}{n}\frac{1}{2^n}\right)$.

Solution : For $a \in \mathbb{R}$ and $x \in (-1, 0]$, we will evaluate

$$\sum_{n=1}^{+\infty} n\left((1+x)^a-1-\binom{a}{1}x-\cdots-\binom{a}{n}x^n\right).$$

For $a \in \mathbb{N}\cup\{0\}$ and $x \in (-1, 0]$, or $a \in \mathbb{R}$ and $x = 0$, the sum trivially equals 0.

Let now $a \in \mathbb{R}\setminus(\mathbb{N}\cup\{0\})$ and $x \in (-1, 0)$. We write

$$\sum_{n=1}^{+\infty} n\left((1+x)^a-1-\binom{a}{1}x-\cdots-\binom{a}{n}x^n\right)=\sum_{n=1}^{+\infty} n\sum_{k=n+1}^{+\infty}\binom{a}{k}x^k.$$

It is easy to observe that for $a \in (m-1, m)$ with $m \in \mathbb{N}$, the summand $\binom{a}{k}x^k$ has sign $\operatorname{sgn}\left((-1)^m\right)$ while $k \geq m$, thus, from Fubini's theorem it is

$$\begin{aligned}\sum_{n=1}^{+\infty} n\sum_{k=n+1}^{+\infty}\binom{a}{k}x^k &= \sum_{n=2}^{+\infty}\left(\sum_{k=1}^{n-1}k\right)\binom{a}{n}x^n\\ &= \frac{1}{2}\sum_{n=2}^{+\infty}n(n-1)\binom{a}{n}x^n\\ &= \frac{1}{2}\left(\sum_{n=2}^{+\infty}n^2\binom{a}{n}x^n-\sum_{n=2}^{+\infty}n\binom{a}{n}x^n\right)\\ &:= \frac{1}{2}\left(A(x)-B(x)\right). \qquad (1)\end{aligned}$$

Now for $x \in (-1,0)$ we have $(1+x)^a = \sum_{n=0}^{+\infty} \binom{a}{n} x^n$ so differentiating we get

$$\begin{aligned}
a(1+x)^{a-1} &= \sum_{n=1}^{+\infty} n x^{n-1}, && \text{multiplying by } x\\
ax(1+x)^{a-1} &= \sum_{n=1}^{+\infty} \binom{a}{n} n x^n, && \text{differentiating again}\\
a(1+x)^{a-2}(1+ax) &= \sum_{n=1}^{+\infty} \binom{a}{n} n^2 x^{n-1}, && \text{and multiplying again by } x\\
ax(1+x)^{a-2}(1+ax) &= \sum_{n=1}^{+\infty} \binom{a}{n} n^2 x^n,
\end{aligned}$$

so

$$\begin{aligned}
A(x) &= ax\left((1+x)^{a-2}(1+ax)-1\right),\\
B(x) &= ax\left((1+x)^{a-1}-1\right)
\end{aligned}$$

and (1) will give

$$\sum_{n=1}^{+\infty} n \sum_{k=n+1}^{+\infty} \binom{a}{k} x^k = \frac{(a-1)ax^2(1+x)^{a-2}}{2}.$$

Thus collecting, for $a \in \mathbb{R}$ and $x \in (-1,0]$ it is

$$\sum_{n=1}^{+\infty} n\left((1+x)^a - 1 - \binom{a}{1}x - \cdots - \binom{a}{n}x^n\right) = \frac{(a-1)ax^2(1+x)^{a-2}}{2}$$

and setting $a = \frac{1}{2}, \quad x = -\frac{1}{2}$ we have

$$\sum_{n=1}^{+\infty} n\left(2^{-1/2} - 1 + \binom{1/2}{1}\frac{1}{2} - \binom{1/2}{2}\frac{1}{4} + \cdots + (-1)^{n+1}\binom{1/2}{n}\frac{1}{2^n}\right) = -2^{-7/2}.$$

Approximation of a logarithmic sum

Proposed problem for MATHPROBLEMS

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The Problem : *Show that*

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{\ln(n+k)} = \frac{1}{2\ln n} + \mathcal{O}(n^{-1}\ln^{-2} n) \qquad n \to +\infty.$$

Solution : We state and prove an elementary Lemma:

Lemma: Let $N \in \mathbb{Z}$. If $f : [N, +\infty) \to \mathbb{R}$ is decreasing, then for every $\mathbb{Z} \ni M > N$ the following inequality holds:

$$\int_N^{M+1} f(x)\,dx \le \sum_{n=N}^{M} f(n) \le f(N) + \int_N^M f(x)\,dx.$$

Proof: Since f is decreasing, for $N+1 \le n \le M$ we have

$$\int_n^{n+1} f(x)\,dx \le \int_n^{n+1} f(n)\,dx = f(n) = \int_{n-1}^{n} f(n)\,dx \le \int_{n-1}^{n} f(x)\,dx$$

so summing from $N+1$ to M we get

$$\int_{N+1}^{M+1} f(x)\,dx \le \sum_{n=N+1}^{M} f(n) \le \int_N^M f(x)\,dx$$

and adding $f(N)$ gives

$$\begin{aligned}\int_N^{M+1} f(x)\,dx &= \int_N^{N+1} f(x)\,dx + \int_{N+1}^{M+1} f(x)\,dx\\ &\leq \int_N^{N+1} f(N)\,dx + \int_{N+1}^{M+1} f(x)\,dx\\ &= f(N) + \int_{N+1}^{M+1} f(x)\,dx\\ &\leq \sum_{n=N}^{M} f(n)\\ &\leq f(N) + \int_N^{M} f(x)\,dx.\end{aligned}$$

For each $n \in \mathbb{N}$, the sum is convergent from Dirichlet's criterion, so we can write:

$$\begin{aligned}\sum_{k=0}^{+\infty} \frac{(-1)^k}{\ln(n+k)} &= \lim_{m\to+\infty} \sum_{k=0}^{2m} \frac{(-1)^k}{\ln(n+k)}\\ &= \sum_{k=0}^{+\infty}\left(\frac{1}{\ln(n+2k)} - \frac{1}{\ln(n+2k+1)}\right). \qquad (1)\end{aligned}$$

Setting $a = n + 2k$, for $k \geq 0$ and as $n \to +\infty$ we have

$$\begin{aligned}\frac{1}{\ln a} - \frac{1}{\ln(a+1)} &= \frac{1}{\ln a}\left(1 - \frac{1}{1 + \frac{\ln(1+a^{-1})}{\ln a}}\right)\\ &= \frac{1}{\ln a}\left(\frac{\ln(1+a^{-1})}{\ln a} + \mathcal{O}\left(\frac{\ln^2(1+a^{-1})}{\ln^2 a}\right)\right)\\ &= \frac{\ln(1+a^{-1})}{\ln^2 a} + \mathcal{O}\left(\frac{\ln^2(1+a^{-1})}{\ln^3 a}\right)\\ &= \frac{1}{a\ln^2 a} + \mathcal{O}(a^{-2}\ln^{-2} a). \qquad (2)\end{aligned}$$

Now from the Lemma, with $f_n(x) = \dfrac{1}{(n+2x)\ln^2(n+2x)}$ and $g_n(x) = \dfrac{1}{(n+2x)^2\ln^2(n+2x)}$ respectively we get that

$$\sum_{k=0}^{+\infty} \frac{1}{a \ln^2 a} = \int_0^{+\infty} \frac{1}{(n+2x)\ln^2(n+2x)}\,dx + \mathcal{O}(n^{-1}\ln^{-2} n)$$

$$\overset{\ln(n+2x)=y}{=\!=\!=\!=\!=} \frac{1}{2\ln n} + \mathcal{O}(n^{-1}\ln^{-2} n) \tag{3}$$

and

$$\sum_{k=0}^{+\infty} \frac{1}{a^2 \ln^2 a} = \int_0^{+\infty} \frac{1}{(n+2x)^2\ln^2(n+2x)}\,dx + \mathcal{O}(n^{-2}\ln^{-2} n)$$

$$= \mathcal{O}(n^{-1}\ln^{-2} n). \tag{4}$$

From (1), (2), (3) and (4) we get the desired result.

Comment: This problem appears as an exercise in [1].

References

[1] N.G. De Bruijn *Asymptotic Methods in Analysis* , Dover Publications Inc., New York, 1981.

A solution to the problem #24 of Volume 1, issue 4, 2010 -2011, of MATHPROBLEMS journal

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• **The Problem :** Proposed by D.M. Bătinetu - Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be sequences of positive real numbers such that

$$\lim_{n\to+\infty}\frac{a_{n+1}}{n^2a_n}=\lim_{n\to+\infty}\frac{b_{n+1}}{n^3b_n}=a>0.$$

Compute $\lim\limits_{n\to+\infty}\sqrt[n+1]{\dfrac{b_{n+1}}{a_{n+1}}}-\sqrt[n]{\dfrac{b_n}{a_n}}$.

• **Solution :** It is well known (look [1] p.46 for example) that z_n being a sequence of positive numbers,

$$\lim_{n\to+\infty}\frac{z_{n+1}}{z_n}=\ell\in\mathbb{R}\Rightarrow\lim_{n\to+\infty}(z_n)^{1/n}=\ell\qquad(*).$$

We set $z_n=\dfrac{b_n}{n^na_n}$, so

$$\frac{z_{n+1}}{z_n}=\frac{b_{n+1}}{n^3b_n}\left(\frac{a_{n+1}}{n^2a_n}\right)^{-1}\left(1+\frac{1}{n}\right)^{-n}\frac{n}{n+1}\to e^{-1}\overset{(*)}{\Rightarrow}(z_n)^{1/n}\to e^{-1}$$

and therefore

$$\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)^n=\frac{b_{n+1}}{n^3b_n}\left(\frac{a_{n+1}}{n^2a_n}\right)^{-1}(z_{n+1})^{-1/(n+1)}\frac{n}{n+1}\to e.$$

Now

$$\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}}-\sqrt[n]{\frac{b_n}{a_n}}=(z_n)^{1/n}\left(\frac{\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}-1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)}\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)^n\right)\to e^{-1},$$

since

$$\begin{aligned}\lim_{n\to+\infty}\frac{\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}-1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)}&=\lim_{n\to+\infty}\frac{\exp\left(\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)\right)-1}{\ln\left(\frac{(n+1)(z_{n+1})^{1/(n+1)}}{n(z_n)^{1/n}}\right)}\\&=\lim_{x\to 0}\frac{e^x-1}{x}=1.\end{aligned}$$

References

[1] W.J. Kaczor, M.T. Nowak *Problems in Mathematical Analysis I, Real Numbers, Sequences and Series* , A.M.S., 2000.

A solution to the problem 30 of Vol.2 Issue.1 2011-2012 of MATHPROBLEMS journal

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The Problem : Proposed by Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.

$$\textit{Evaluate:}\qquad \lim_{x\to 0}\int_{2011x}^{2012x}\frac{\sin^n t}{t^m}\,dt \qquad \textit{where } n,m\in\mathbb{N}.$$

Solution : More generally, let $0<a<b$ and $n,m\in\mathbb{N}$.

1. For $n-m\geq -1$ we have

$$\begin{aligned}
\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt &= \int_{ax}^{bx} t^{n-m}\left(1+\mathcal{O}(t^2)\right)^n \\
&= \int_{ax}^{bx} t^{n-m}\left(1+\mathcal{O}(t^2)\right) \\
&= \int_{ax}^{bx} t^{n-m}+\mathcal{O}\left(t^{n-m+2}\right)\,dt \\
&= \begin{cases} \left.\dfrac{t^{n-m+1}}{n-m+1}\right|_{ax}^{bx}+\mathcal{O}\left(\left.\dfrac{t^{n-m+3}}{n-m+3}\right|_{ax}^{bx}\right) & ,n-m\geq 0 \\[2ex] \ln|t|\Big|_{ax}^{bx}+\mathcal{O}\left(t^2\Big|_{ax}^{bx}\right) & ,n-m=-1 \end{cases} \\
&= \begin{cases} \dfrac{b^{n-m+1}-a^{n-m+1}}{n-m+1}x^{n-m+1}+\mathcal{O}(x^{n-m+3}) & ,n-m\geq 0 \\[2ex] \ln\dfrac{b}{a}+\mathcal{O}(x^2) & ,n-m=-1 \end{cases} \\
&\xrightarrow{x\to 0} \begin{cases} 0 & ,n-m\geq 0 \\ \ln\dfrac{b}{a} & ,n-m=-1 \end{cases}.
\end{aligned}$$

$$\text{Since} \qquad \int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt \xlongequal{t=-y} (-1)^{n-m+1}\int_{a(-x)}^{b(-x)}\frac{\sin^n y}{y^m}\,dy, \tag{1}$$

2. For $n-m\leq -2$:

- If $n-m$ is odd, then for some $0<\varepsilon<1$ and while $x\to 0^+$ we have

$$\begin{aligned}
&(1-\varepsilon)\leq\frac{\sin t}{t}\leq 1\\
\Rightarrow&\frac{(1-\varepsilon)^n}{t^{m-n}}\leq\frac{\sin^n t}{t^m}\leq\frac{1}{t^{m-n}}\\
\Rightarrow&(1-\varepsilon)^n\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}}\leq\int_{ax}^{bx}\frac{\sin^n t}{t^m}\leq\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}},
\end{aligned}$$

thus $\lim\limits_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=+\infty$ and from (1) $\lim\limits_{x\to 0^-}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=\lim\limits_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=+\infty.$

- If $n-m$ is even, then similarly while $x\to 0^+$ we have

$$(1-\varepsilon)^n\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}}\leq\int_{ax}^{bx}\frac{\sin^n t}{t^m}\leq\frac{b^{n-m+1}-a^{n-m+1}}{(n-m+1)x^{m-n-1}},$$

thus $\lim\limits_{x\to 0^+}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=+\infty$ and from (1) $\lim\limits_{x\to 0^-}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=\lim\limits_{x\to 0^+}-\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt=-\infty$ and the limit doesn't exist.

Collecting we have

$$\lim_{x\to 0}\int_{ax}^{bx}\frac{\sin^n t}{t^m}\,dt\begin{cases}=0 & ,n-m\geq 0\\ =\ln\dfrac{b}{a} & ,n-m=-1\\ =+\infty & ,n-m\leq -2\quad\text{and}\quad n-m=\text{odd}\\ \text{doesn't exist} & ,n-m\leq -2\quad\text{and}\quad n-m=\text{even}\end{cases}.$$

A solution to the problem 33 of Vol.2 Issue.1 2011-2012 of MATHPROBLEMS journal

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The Problem : Proposed by Ovidiu Furdui, Cluj, Romania.

Find the value of $\lim\limits_{n\to+\infty}\int_0^{\pi/2}\sqrt[n]{\sin^n x+\cos^n x}\,dx.$

Solution : We set $f_n(x):=\sqrt[n]{\sin^n x+\cos^n x}$, so

- $f_n\left(\dfrac{\pi}{2}\right)=1\xrightarrow{n\to+\infty}1$ and
- for $x\in\left[0,\dfrac{\pi}{2}\right)$, since $f_n(x)=\cos x\exp\left(\dfrac{\ln\left(1+\tan^n x\right)}{n}\right)$,

$$\begin{aligned}
x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)&\Rightarrow\lim_{n\to+\infty}\frac{\ln\left(1+\tan^n x\right)}{n}\overset{\text{DLH}}{=\!=}\lim_{n\to+\infty}\frac{\ln\left(\tan x\right)}{1+\tan^{-n}x}=\ln\left(\tan x\right)\\
&\Rightarrow f_n(x)\xrightarrow{n\to+\infty}\sin x \qquad\qquad \text{and}\\
x\in\left[0,\frac{\pi}{4}\right]&\Rightarrow\lim_{n\to+\infty}\frac{\ln\left(1+\tan^n x\right)}{n}=0,
\end{aligned}$$

so

$$f_n(x)\xrightarrow{n\to+\infty}\begin{cases}\sin x & ,x\in\left(\dfrac{\pi}{4},\dfrac{\pi}{2}\right]\\[2ex] \cos x & ,x\in\left[0,\dfrac{\pi}{4}\right]\end{cases}.$$

Furthermore, it is $|f_n(x)|\leq 2$, so by the Dominated Convergence theorem we have

$$\lim_{n\to+\infty}\int_0^{\pi/2}\sqrt[n]{\sin^n x+\cos^n x}\,dx=\int_0^{\pi/4}\cos x\,dx+\int_{\pi/4}^{\pi/2}\sin x\,dx=\sqrt{2}.$$

A solution to the problem 41 of Vol.2 Issue.2 2011-2012 of MATHPROBLEMS journal

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The Problem : Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.

$$\textit{Evaluate} \quad \int_0^{\pi/2} 4\cos^2 x \ln^2(\cos x)\, dx.$$

Solution : At first, using two basic representations of the Digamma function Ψ, i.e.:

$$\Psi(x) = (\ln\Gamma(x))' = \frac{\Gamma'(x)}{\Gamma(x)} \qquad x \in \mathbb{R} \setminus -\mathbb{N} \tag{1}$$

$$\Psi(x) = -\gamma + \sum_{k=1}^{+\infty}\left(\frac{1}{k} - \frac{1}{x+k-1}\right) \qquad x \in \mathbb{R} \setminus -\mathbb{N}, \tag{2}$$

we get some special values for Ψ and Ψ'.

From (2) we have that for $x \in \mathbb{R} \setminus -\mathbb{N}$:

$$\Psi(x+1) - \Psi(x) = \frac{1}{x} - \lim_{n\to+\infty}\frac{1}{x+n} = \frac{1}{x}. \tag{3}$$

Furthermore, differentiating (2), [1] gives

$$\Psi'(x) = \sum_{k=1}^{+\infty}\frac{1}{(x+k-1)^2} \qquad x > 0. \tag{4}$$

[1] for $k \geq 1$, $\dfrac{1}{k} - \dfrac{1}{x+k-1}$ has a continuous derivative and $\displaystyle\sum_{k=1}^{+\infty}\frac{1}{(x+k-1)^2}$ converges uniformly on $[a,b]$ with $a > 0$

On account of the above we have

$$
\begin{aligned}
\Psi(1) &\overset{(2)}{=} -\gamma \\
\Psi(2) &\overset{(3)}{=} \Psi(1)+1 = 1-\gamma \\
\Psi\left(\frac{3}{2}\right) &\overset{(3)}{=} 2+\Psi\left(\frac{1}{2}\right) \overset{(2)}{=} 2-\gamma+2\sum_{k=1}^{+\infty}\left(\frac{1}{2k}-\frac{1}{2k-1}\right) = 2-\gamma-2\ln 2. \\
\Psi'(2) &\overset{(4)}{=} \sum_{k=1}^{+\infty}\frac{1}{(k+1)^2} = \frac{\pi^2}{6}-1 \\
\Psi'\left(\frac{3}{2}\right) &\overset{(4)}{=} 4\sum_{k=1}^{+\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{2}-4.
\end{aligned}
$$

Now, for the given integral we have

$$\int_0^{\pi/2} 4\cos^2 x\ln^2(\cos x)\,dx \overset{\cos x = t}{=} 4\int_0^1 \frac{t^2\ln^2 t}{\sqrt{1-t^2}}\,dt.$$

But

$$\int_0^1 \frac{t^{a-1}}{\sqrt{1-t^2}}\,dt \overset{t^2=x}{=} \frac{1}{2}B\left(\frac{a}{2},\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a}{2}\right)}{2\Gamma\left(\frac{a+1}{2}\right)}$$

and by Leibniz's rule, differentiating twice under the integral sign and using (1) we get

$$\int_0^1 \frac{t^{a-1}\ln^2 t}{\sqrt{1-t^2}}\,dt = \frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}\left(\Psi^2\left(\frac{a}{2}\right)-2\Psi\left(\frac{a+1}{2}\right)\Psi\left(\frac{a}{2}\right)+\Psi^2\left(\frac{a+1}{2}\right)+\Psi'\left(\frac{a}{2}\right)-\Psi'\left(\frac{a+1}{2}\right)\right).$$

For $a=3$ and from the special values of Ψ and Ψ' evaluated above we have that

$$
\begin{aligned}
\int_0^{\pi/2} 4\cos^2 x\ln^2(\cos x)\,dx &= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\left(\Psi^2\left(\frac{3}{2}\right)-2\Psi(2)\,\Psi\left(\frac{3}{2}\right)+\Psi^2(2)+\Psi'\left(\frac{3}{2}\right)-\Psi'(2)\right) \\
&= \frac{\pi^3}{12}+\pi\ln^2 2-\pi\ln 2-\frac{\pi}{2}.
\end{aligned}
$$

A solution to the problem 35 (Mathcontest Section)

of Vol.2 Issue 3 2011-2012 of MATHPROBLEMS journal

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July 22, 2012

The Problem : *Compute* $\lim\limits_{n\to+\infty}\frac{1}{n^3}\int_0^n\frac{n^2+x^2}{5^{-x}+7}\,dx.$

Solution : We have

$$\frac{1}{n^3}\int_0^n\frac{n^2+x^2}{5^{-x}+7}\,dx \xlongequal{ny=x} \int_0^1\frac{1+y}{5^{-ny}+7}\,dy.$$

But

$$\frac{1+y}{5^{-ny}+7}\to\begin{cases}\dfrac{1+y}{7} & ,y\in(0,1]\\[2ex] \dfrac{1}{8} & ,y=0\end{cases}\qquad\text{and for}\quad y\in[0,1],\quad\left|\frac{1+y}{5^{-ny}+7}\right|\leq\frac{1+y}{7}$$

which is integrable, so by the Dominated Convergence Theorem

$$\lim_{n\to+\infty}\frac{1}{n^3}\int_0^n\frac{n^2+x^2}{5^{-x}+7}\,dx=\int_0^1\frac{1+y}{7}\,dy=\frac{3}{14}.$$

A solution to the problem 43 of Vol.2 Issue 3 2011-2012 of MATHPROBLEMS journal

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July 22, 2012

The Problem : Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National Colege, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Jointly).

Let a be a positive real number and $\Gamma(x)$ be the Gamma function (or Euler's second integral). Calculate

$$\lim_{x\to+\infty}\left((x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)-x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)\right).$$

Solution : From Stirling's formula we have

$$\Gamma(x+1)=\frac{\sqrt{2\pi}x^{x+1/2}}{e^x}\left(1+\mathcal{O}(x^{-1})\right)\qquad x\to+\infty,$$

so

$$\ln\left(\Gamma(x+1)\right)=x\ln x-x+\frac{\ln x}{2}+\frac{\ln 2\pi}{2}+\mathcal{O}(x^{-1})\qquad\text{and}$$

$$\begin{aligned}\left(\Gamma(x+1)\right)^{\frac{1}{x}}&=\exp\left(\ln x-1+\frac{\ln x}{2x}+\frac{\ln 2\pi}{2x}+\mathcal{O}(x^{-2})\right)\\&=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).\end{aligned}\tag{1}$$

For $x+1$ in (1) instead of x we get

$$\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).$$

Furthermore,

$$\begin{aligned}x\sin\left(\frac{1}{x}\right)&=1+\mathcal{O}(x^{-2}),\\(x+a)\sin\left(\frac{1}{x+a}\right)&=1+\mathcal{O}(x^{-2})\end{aligned}$$

and therefore

$$x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\mathcal{O}\left(x^{-1}\ln^2 x\right)\qquad\text{and}$$

$$(x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).$$

On account of the above

$$(x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)-x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)=\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right)\to\frac{1}{e}.$$

A solution to the problem 44 of Vol.2 Issue 3 2011-2012 of MATHPROBLEMS journal

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July 23, 2012

The Problem : Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

Let A *denote the Glaisher-Kinkelin constant defined by*

$$A = \lim_{n\to+\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.282427130\dots .$$

Prove that

$$\sum_{p=1}^{+\infty} \frac{\zeta(2p+1)-1}{p+2} = -\frac{\gamma}{2} - 6\ln A + \ln 2 + \frac{7}{6},$$

where ζ *is the Riemann zeta function defined by* $\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{n^s}$ *for* $\Re(s) > 1$.

Solution : At first we observe that for $|x| < 1$ it is

$$\begin{aligned}\sum_{p\geq 1} \frac{x^{2p+1}}{p+2} &= \frac{1}{x^3} \sum_{p\geq 3} \frac{\left(x^2\right)^p}{p} = -\frac{1}{x^3}\left(\ln(1-x^2) + x^2 + \frac{x^4}{2}\right) \\ &= -\frac{x}{2} - \frac{1}{x} - \frac{\ln(1-x^2)}{x^3}. \qquad (1)\end{aligned}$$

Now we have

$$\begin{aligned}
\sum_{p=1}^{+\infty}\frac{\zeta(2p+1)-1}{p+2} &= \sum_{p\geq 1}\frac{1}{p+2}\left(\sum_{n\geq 1}\frac{1}{n^{2p+1}}-1\right)\\
&= \sum_{p\geq 1}\sum_{n\geq 2}\frac{\left(n^{-1}\right)^{2p+1}}{p+2}\\
{}^{1}&= \sum_{n\geq 2}\sum_{p\geq 1}\frac{\left(n^{-1}\right)^{2p+1}}{p+2}\\
&\overset{(1)}{=\!=} -\sum_{n\geq 2}\left(\frac{1}{2n}+n+n^3\ln\left(1-\frac{1}{n^2}\right)\right)\\
&= -\lim_{N\to+\infty}\sum_{n=2}^{N}\left(\frac{1}{2n}+n+n^3\ln\left(1-\frac{1}{n^2}\right)\right)\\
&= -\lim_{N\to+\infty}\left(\frac{H_N-1}{2}+\frac{(N+2)(N-1)}{2}+\ln\left(\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^3}\right)\right)\\
:&= -\lim_{N\to+\infty}\left(\frac{H_N}{2}+\frac{N^2}{2}+\frac{N}{2}-\frac{3}{2}+\ln A_N\right). \qquad (2)
\end{aligned}$$

But

$$\begin{aligned}
A_N &= \prod_{n=2}^{N}\frac{(n-1)^{n^3}}{n^{n^3}}\prod_{n=2}^{N}\frac{(n+1)^{n^3}}{n^{n^3}}\\
&= \frac{1}{N^{N^3}}\prod_{k=2}^{N-1}k^{(k+1)^3-k^3}\cdot\frac{(N+1)^{N^3}}{2^{2^3}}\prod_{k=3}^{N}k^{(k-1)^3-k^3}\\
&= \frac{1}{2}\cdot\frac{(N+1)^{N^3}}{N^{(N+1)^3}}\prod_{k=1}^{N}k^{6k}\\
&= \left(N^{-N^2/2-N/2-1/12}e^{N^2/4}\prod_{k=1}^{N}k^k\right)^6\cdot\frac{(N+1)^{N^3}N^{3N^2+3N+1/2}}{2N^{(N+1)^3}e^{3N^2/2}}, \qquad (3)
\end{aligned}$$

and since

$$H_N = \ln N+\gamma+o(1) \qquad \text{and}$$
$$\ln\frac{(N+1)^{N^3}N^{3N^2+3N+1/2}}{2N^{(N+1)^3}e^{3N^2/2}} = -\frac{N^2}{2}-\frac{N}{2}-\frac{\ln N}{2}+\frac{1}{3}-\ln 2+o(1),$$

on account of (3), (2) will give

[1] the summands are positive

$$\sum_{p=1}^{+\infty}\frac{\zeta(2p+1)-1}{p+2}=\lim_{N\to+\infty}\left(-\frac{\gamma}{2}-6\ln A+\ln 2+\frac{7}{6}+o(1)\right)$$

which gives the desired result.

A solution to the problem 51 of Vol.2 Issue 4 2012 of MATHPROBLEMS journal

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December 22, 2012

The Problem. *Proposed by D.M. Batineţu-Giurgiu, Matei Basarab National Colege, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania (Jointly).*

Let $a \in (1, \infty)$ *and* $b \in (0, \infty)$. *Calculate*

$$\lim_{n \to +\infty} n \left(2 - \exp \left(\sum_{k=1}^{n} \frac{(n+k)^{a-1}}{(n+k)^a + b} \right) \right).$$

Solution : We use that

$$H_n = \ln n + \gamma + \frac{1}{2n} + \mathcal{O}(n^{-2}) \qquad n \to +\infty. \tag{1}$$

Note that for $m > 1$, it is

$$\sum_{k \geq n+1} \frac{1}{k^m} = \mathcal{O}(n^{1-m}), \tag{2}$$

since from Cesàro Stolz we have

$$\lim_{n \to +\infty} n^{m-1} \sum_{k \geq n+1} \frac{1}{k^m} = \lim_{n \to +\infty} \frac{-(n+1)^{-m}}{(n+1)^{1-m} - n^{1-m}} = \lim_{n \to +\infty} \left(m - 1 + \mathcal{O}(n^{-1}) \right)^{-1} \to \frac{1}{m-1}.$$

Now for $a > 1$ and $b \in \mathbb{R}$,

$$
\begin{aligned}
S_{a,b,n} &:= \sum_{k=1}^{n} \frac{(n+k)^{a-1}}{(n+k)^a + b} = \sum_{k=1}^{n} \frac{1}{n+k}\left(1 - \frac{b}{(n+k)^a} + \mathcal{O}(n^{-2a})\right) \\
&= H_{2n} - H_n - b\sum_{k=1}^{n} \frac{1}{(n+k)^{a+1}} + \mathcal{O}(n^{-2a}) \\
&\overset{(1),a>1}{=\!=\!=} \ln 2 - \frac{1}{4n} - b\sum_{k=1}^{n} \frac{1}{(n+k)^{a+1}} + \mathcal{O}(n^{-2}) \\
&= \ln 2 - \frac{1}{4n} - b\left(\sum_{k=1}^{2n} \frac{1}{k^{a+1}} - \sum_{k=1}^{n} \frac{1}{k^{a+1}}\right) + \mathcal{O}(n^{-2}) \\
&= \ln 2 - \frac{1}{4n} - b\left(\sum_{k\geq n+1} \frac{1}{k^{a+1}} - \sum_{k\geq 2n+1} \frac{1}{k^{a+1}}\right) + \mathcal{O}(n^{-2}) \\
&\overset{(2)}{=} \ln 2 - \frac{1}{4n} + \mathcal{O}\left(n^{-\min\{a,2\}}\right).
\end{aligned}
$$

On account of the above

$$
n\,(2 - \exp S_{a,b,n}) = n\left(2 - 2\left(1 - \frac{1}{4n} + \mathcal{O}(n^{-\min\{a,2\}})\right)\right) = \frac{1}{2} + \mathcal{O}(n^{-\min\{a-1,1\}}) \xrightarrow{a>1} \frac{1}{2}.
$$

A solution to the problem 53 of Vol.2 Issue 4 2012 of MATHPROBLEMS journal

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December 21, 2012

The Problem. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

The Stirling numbers of the first kind denoted by $s(n,k)$ *are the special numbers defined by the generating function*

$$z(z-1)(z-2)\cdots(z-n+1)=\sum_{k=0}^{n}s(n,k)z^k.$$

Let n *and* m *be nonnegative integers with* $n>m-1$. *Prove that*

$$\int_0^1\frac{\ln^n x}{(1-x)^m}\,dx=\begin{cases}(-1)^n n!\zeta(n+1) & ,m=1\\ (-1)^{n+m-1}\dfrac{n!}{(m-1)!}\displaystyle\sum_{i=1}^{m-1}(-1)^i s(m-1,i)\zeta(n+1-i) & ,m\geq 2\end{cases}.$$

Solution : By repeated integration by parts one can easily show that for k a non negative integer and n a positive integer we have

$$\int x^k\ln^n x\,dx=x^{k+1}\left(\frac{\ln^n x}{k+1}-\frac{n\ln^{n-1}x}{(k+1)^2}+\frac{n(n-1)\ln^{n-2}x}{(k+1)^3}-\ldots+\frac{(-1)^n n!}{(k+1)^{n+1}}\right)+c. \tag{1}$$

Furthermore, we will use that

$$(1-x)^{-m}=\sum_{k\geq 0}\binom{k+m-1}{k}x^k\qquad x\in(-1,1). \tag{2}$$

Now :

$$\begin{aligned}\int_0^1\frac{\ln^n x}{(1-x)^m}\,dx&\overset{(2)}{=}\int_0^1\sum_{k\geq 0}\binom{k+m-1}{k}x^k\ln^n x\,dx^1=\sum_{k\geq 0}\binom{k+m-1}{k}\int_0^1 x^k\ln^n x\,dx\\&\overset{(1)}{=}\sum_{k\geq 0}\binom{k+m-1}{k}\frac{(-1)^n n!}{(k+1)^{n+1}}\\&{}^2=(-1)^n n!\sum_{k\geq 0}\frac{\binom{k+m-1}{m-1}}{(k+1)^{n+1}}{}^3=(-1)^{n+m-1}n!\sum_{k\geq 0}\frac{\binom{-k-1}{m-1}}{(k+1)^{n+1}}\end{aligned}$$

But for $m \geq 2$, from the definition of Stirling numbers, we have

$$\binom{-k-1}{m-1} = \frac{(-k-1)(-k-1-1)(-k-1-2)\cdots(-k-1-(m-1-1))}{(m-1)!} = \frac{1}{(m-1)!}\sum_{i=0}^{m-1} s(m-1,i)(-k-1)^i,$$

and for $m = 1$, $\binom{-k-1}{m-1} = 1$, so,

$$\int_0^1 \frac{\ln^n x}{1-x}\,dx = (-1)^n n!\zeta(n+1)$$

and for $m \geq 2$:

$$\begin{aligned}\int_0^1 \frac{\ln^n x}{(1-x)^m}\,dx &= (-1)^{n+m-1}\frac{n!}{(m-1)!}\sum_{k\geq 0}\sum_{i=0}^{m-1}\frac{(-1)^i s(m-1,i)}{(k+1)^{n-i+1}} \\ &\overset{4}{=} (-1)^{n+m-1}\frac{n!}{(m-1)!}\sum_{i=0}^{m-1}(-1)^i s(m-1,i)\sum_{k\geq 0}\frac{1}{(k+1)^{n-i+1}} \\ &\overset{5}{=} (-1)^{n+m-1}\frac{n!}{(m-1)!}\sum_{i=1}^{m-1}(-1)^i s(m-1,i)\zeta(n-i+1)\end{aligned}$$

and we get what we wanted. $\square$

[1]for fixed n, $x^k \ln^n x$ is either non positive or non negative for $x \in [0,1]$.

[2] $\binom{n}{k} = \binom{n}{n-k}$, n non negative integer and $k \in \mathbb{Z}$

[3] $\binom{r}{k} = \binom{-r+k-1}{k}(-1)^k, \qquad k \in \mathbb{Z}$

[4]note that, since $s(m-1,i)$ is the coefficient of z^i in $z(z-1)(z-2)\cdots(z-m+2)$, from Vieta's formulas $\operatorname{sgn}(s(m-1,i)) = (-1)^{m-i+1}$, so, for fixed m, $(-1)^i s(m-1,i)$ preserves its sign

[5]$z(z-1)(z-2)\cdots(z-m-2)$ has zero constant term.

A solution to the problem 54 of Vol.2 Issue 4 2012 of MATHPROBLEMS journal

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December 22, 2012

The Problem. *Proposed by Moubinool Omarjee, Paris, France.*

Let $f : [0, +\infty) \to \mathbb{R}$ *be a measurable function such that* $g(t) = e^t f(t) \in L^1(\mathbb{R}_+)$*; the space of Lebesgue integrable functions. Find*

$$\lim_{n\to+\infty} \int_{\mathbb{R}_+} f(t) \left(4\sinh\frac{nt+1}{2}\sinh\frac{nt-1}{2}\right)^{1/n} dt$$

where $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$.

Solution : At first we note that $4\sinh\dfrac{nt+1}{2}\sinh\dfrac{nt-1}{2} \geq 0 \Leftrightarrow t \geq \dfrac{1}{n}$, so we assume that the integrand is $f(t)h(t)^{1/n}$ where $h(t) = \left(4\sinh\dfrac{nt+1}{2}\sinh\dfrac{nt-1}{2}\right)^+$, i.e. $h(t) = \left(4\sinh\dfrac{nt+1}{2}\sinh\dfrac{nt-1}{2}\right)$ for $t \geq \dfrac{1}{n}$ and 0 for $t \in [0, 1/n)$.

Now keeping the above notation, for $t \geq 0$ we have

$$h(t)^{1/n} = \left(\left(2\cosh(nt) - e - \frac{1}{e}\right)^+\right)^{1/n} \leq e^t \Leftarrow e^{-nt} \leq e + \frac{1}{e}$$

which is true since $e^{-nt} \leq 1 \leq e + \dfrac{1}{e}$.

Furthermore, since $g(t) \in L^1(\mathbb{R}_+)$, it is $f(t)^- e^t, f(t)^+ e^t \in L^1(\mathbb{R}_+)$, so

$$\left|f(t)h(t)^{1/n}\right| \leq |f(t)e^t| = (f(t)^+ - f(t)^-)e^t \in L^1(\mathbb{R}_+)$$

and considering the fact that $h(t)^{1/n} \to e^t$ for $t > 0$ we get

$$\lim_{n\to+\infty} \int_{\mathbb{R}_+} f(t) \left(4\sinh\frac{nt+1}{2}\sinh\frac{nt-1}{2}\right)^{1/n} dt = \int_{\mathbb{R}_+} f(t)e^t\, dt$$

by Lebesgue's Dominated Convergence Theorem.

A solution to the problem #152 of Missouri State University's Advanced Problem Page

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Thursday 20/10/2011

• **The Problem :** Evaluate $\sum_{k=1}^{+\infty}(-1)^k\frac{\ln k}{k}$.

◇ We set $S_n := \sum_{k=1}^{n}(-1)^k\frac{\ln k}{k}$ and observe that the series converge from Dirichlet's criterion, since $(-1)^k$ has bounded partial sums and $\frac{\ln k}{k}$ is finally strictly decreasing to 0.
Hence $\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k} = \lim_{n\to+\infty} S_{2n}$.
Now:

$$S_{2n} = \sum_{k=1}^{n}\frac{\ln 2k}{2k} - \sum_{k=1}^{n}\frac{\ln(2k-1)}{2k-1} = \frac{\ln 2}{2}\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{2}\sum_{k=1}^{n}\frac{\ln k}{k} - \left(\sum_{k=1}^{2n}\frac{\ln k}{k} - \sum_{k=1}^{n}\frac{\ln 2k}{2k}\right) =$$

$$\ln 2H_n + \sum_{k=1}^{n}\frac{\ln k}{k} - \sum_{k=1}^{2n}\frac{\ln k}{k} = \ln 2H_n - \sum_{k=1}^{n}\frac{\ln(n+k)}{n+k} = \ln 2H_n - \sum_{k=1}^{n}\frac{\ln n + \ln(1+k/n)}{n+k} =$$

$$\ln 2H_n - \ln n(H_{2n}-H_n) - \frac{1}{n}\sum_{k=1}^{n}\frac{\ln(1+k/n)}{1+k/n} =$$

$$H_n\ln(2n) - H_{2n}\ln n - \frac{1}{n}\sum_{k=1}^{n}\frac{\ln(1+k/n)}{1+k/n} \overset{H_n=\ln n+\gamma+\mathcal{O}(1/n)}{=}$$

$$\gamma\ln 2 + \mathcal{O}(1/n) - \frac{1}{n}\sum_{k=1}^{n}\frac{\ln(1+k/n)}{1+k/n} \to \gamma\ln 2 - \int_0^1\frac{\ln(1+x)}{1+x}\,dx = \gamma\ln 2 - \frac{\ln^2 2}{2}.$$

A solution to the problem #153 of Missouri State University's Advanced Problem Page

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Tuesday 14/11/2011

• **The Problem :** Evaluate the integral $\int_0^1 [-\ln x]\,dx$, where $[x]$ is the greatest integer less than or equal to x.

• **Solution :** At first we prove an elementary Lemma.

Lemma : Let $a_n = a_1 + (n-1)a$ and $b_n = b_1 b^{n-1}$ with $a, a_1, b_1 \in \mathbb{R}$, $b \neq 1$ be an arithmetic and a geometric progression respectively. If $c_n := a_n b_n$, then

$$\sum_{k=1}^{n} c_k = \frac{a_1 b_1 (1-b^n)}{1-b} + \frac{ab_1 b}{(1-b)^2}(1 - nb^{n-1} + (n-1)b^n).$$

Proof : We have

$$\begin{aligned}
\sum_{k=1}^{n} c_n &= \sum_{k=1}^{n} (a_1 + (k-1)a) b_1 b^{k-1} \\
&= a_1 b_1 \sum_{k=1}^{n} b^{k-1} + ab_1 b \sum_{k=1}^{n-1} k b^{k-1} \\
&= \frac{a_1 b_1 (1-b^n)}{1-b} + ab_1 b \left(\sum_{k=1}^{n-1} b^k \right)' \\
&= \frac{a_1 b_1 (1-b^n)}{1-b} + ab_1 b \left(\frac{b - b^n}{1-b} \right)' \\
&= \frac{a_1 b_1 (1-b^n)}{1-b} + \frac{ab_1 b}{(1-b)^2}(1 - nb^{n-1} + (n-1)b^n).
\end{aligned}$$

Now since for $k \in \mathbb{N}_0$ it is

$$k \leq -\ln x < k+1 \Leftrightarrow e^{-(k+1)} < x \leq e^{-k}, \text{ we can write}$$

$$
\begin{aligned}
\int_0^1 [-\ln x]\,dx &= \sum_{k=0}^{+\infty} \int_{e^{-(k+1)}}^{e^{-k}} [-\ln x]\,dx \\
&= \sum_{k=0}^{+\infty} \int_{e^{-(k+1)}}^{e^{-k}} k\,dx \\
&= \sum_{k=0}^{+\infty} k\left(e^{-k} - e^{-(k+1)}\right) \\
&= (1-e^{-1}) \sum_{k=0}^{+\infty} ke^{-k} \\
&= (1-e^{-1}) \lim_{m\to+\infty} \sum_{k=0}^{m} ke^{-k} \\
&\overset{(*)}{=} (1-e^{-1}) \lim_{m\to+\infty} \frac{e^{-1}}{(1-e^{-1})^2}\left(1-(n-1)\left(e^{-1}\right)^{n-2} + (n-2)\left(e^{-1}\right)^{n-1}\right) \\
&= (1-e^{-1}) \frac{e^{-1}}{(1-e^{-1})^2} = \frac{1}{e-1}.
\end{aligned}
$$

$(*)$ By the Lemma with $a_n = n$ and $b_n = e^{-n}$.

A solution to the problem #158 of
Missouri State University's Advanced Problem Page
Three Infinite Alternating Series

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September 26, 2012

The Problem : Find a closed form for each of the following alternating infinite series:

$$\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\cdots,$$

$$\frac{1}{1\cdot 2\cdot 3}-\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}-\frac{1}{4\cdot 5\cdot 6}+\cdots,$$

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}-\frac{1}{2\cdot 3\cdot 4\cdot 5}+\frac{1}{3\cdot 4\cdot 5\cdot 6}-\frac{1}{4\cdot 5\cdot 6\cdot 7}+\cdots.$$

Solution : Using the notation $\Gamma(k+1)=k!$ for $k\in\mathbb{N}\cup\{0\}$, where Γ is the Gamma function, we will, more generally, show that

$$\sum_{k\geq 1}(-1)^{k-1}\frac{1}{k\cdot(k+1)\cdots(k+m)}=\frac{2^m}{\Gamma(m+1)}\left(\ln 2-\sum_{k=1}^{m}\frac{(1/2)^k}{k}\right),\qquad m\in\mathbb{N}.$$

Using the identity $\Gamma(k+1)=k\Gamma(k),\quad k>0$, by a direct calculation we see that

$$\frac{\Gamma(k)}{\Gamma(k+m+1)}=\frac{1}{m}\left(\frac{\Gamma(k)}{\Gamma(k+m)}-\frac{\Gamma(k+1)}{\Gamma(k+m+1)}\right),\qquad m\in\mathbb{N}. \tag{1}$$

Now setting $A_{m,n}:=\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k\cdot(k+1)\cdots(k+m)},\quad m\in\mathbb{N}\cup\{0\}$, on account of (1) we have:

$$\begin{aligned}
A_{m,n} &= \sum_{k=1}^{n}(-1)^{k-1}\frac{\Gamma(k)}{\Gamma(k+m+1)}\\
&= \frac{1}{m}\sum_{k=1}^{n}(-1)^{k-1}\left(\frac{\Gamma(k)}{\Gamma(k+m)}-\frac{\Gamma(k+1)}{\Gamma(k+m+1)}\right)\\
&= \frac{1}{m}\left(\frac{1}{\Gamma(m+1)}+(-1)^{n}\frac{\Gamma(n+1)}{\Gamma(n+m+1)}+2\sum_{k=2}^{n}(-1)^{k-1}\frac{\Gamma(k)}{\Gamma(k+m)}\right)\\
&= \frac{1}{m}\left(\frac{1}{\Gamma(m+1)}+(-1)^{n}\frac{\Gamma(n+1)}{\Gamma(n+m+1)}+2\left(A_{m-1,n}-\frac{1}{\Gamma(m+1)}\right)\right),
\end{aligned}$$

so

$$A_{m,n} = \frac{1}{m}\left(-\frac{1}{\Gamma(m+1)}+(-1)^{n}\frac{\Gamma(n+1)}{\Gamma(n+m+1)}+2A_{m-1,n}\right),$$

and, since for $m \in \mathbb{N}\cup\{0\}$ by Dirichlet's criterion $\displaystyle\sum_{k\geq 1}(-1)^{k-1}\frac{\Gamma(k)}{\Gamma(k+m+1)}$ converges and for $m\in\mathbb{N}$ we have $\displaystyle(-1)^{n}\frac{\Gamma(n+1)}{\Gamma(n+m+1)}\to 0$, setting $\displaystyle A_m := \sum_{k\geq 1}(-1)^{k-1}\frac{\Gamma(k)}{\Gamma(k+m+1)}$ and letting $n\to+\infty$ we have

$$A_m = -\frac{1}{m\Gamma(m+1)}+\frac{2}{m}A_{m-1}, \qquad m\in\mathbb{N}. \tag{2}$$

Since $A_0 = \ln 2$, with a simple inductive argument, (2) yields

$$A_m = \frac{2^m}{\Gamma(m+1)}\left(\ln 2-\sum_{k=1}^{m}\frac{(1/2)^k}{k}\right), \qquad m\in\mathbb{N}.$$

A solution to the problem H -706 of Volume 49 Number 3, August 2011 of The Fibonacci Quarterly

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Wednesday 9/11/2011

• **The Problem :** If $S_n = \frac{1}{2}\left(\sum_{k=n+1}^{3n} \frac{1}{k^2-n^2}\right)^{-1}$, show that $S_n \sim \pi(n)$ as $n \to +\infty$, where $\pi(n)$ is the counting function of the primes $p \le n$.

• **Solution :** By the restricted form of the Euler Maclaurin summation formula, [1]

if $f : [a,b] \to \mathbb{R}$ where $a, b \in \mathbb{N}$ is continuously differentiable on $[a,b]$, then

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \int_a^b \left(\{x\} - \frac{1}{2}\right) f'(x)\,dx$$

where $\{\cdot\}$ denotes the fractional part.

We apply this on $f_n(x) = \frac{1}{x^2-n^2}$ on $[n+1, 3n]$ with $f_n^{'}(x) = -\frac{2x}{(x^2-n^2)^2} < 0$, so we have

$$\begin{aligned}
\sum_{k=n+1}^{3n} \frac{1}{n^2-k^2} &= \int_{n+1}^{3n} \frac{1}{x^2-n^2}\,dx + \frac{f_n(3n)+f_n(n+1)}{2} + \int_{n+1}^{3n} \left(\{x\}-\frac{1}{2}\right) f_n^{'}(x)\,dx \\
&= \frac{1}{2n}\ln\left(\frac{n+1}{2}\right) + \frac{8n^2+2n+1}{32n^3+16n^2} - \int_{n+1}^{3n} \left(\{x\}-\frac{1}{2}\right) f_n^{'}(x)\,dx \\
&= \frac{\ln n}{2n} + \mathcal{O}(n^{-1}), \qquad \text{since}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2n}\ln\left(\frac{n+1}{2}\right) &= \frac{\ln n}{2n} + \mathcal{O}(n^{-1}), \\
\frac{8n^2+2n+1}{32n^3+16n^2} &= \mathcal{O}(n^{-1}), \qquad \text{and} \\
\left|\int_{n+1}^{3n} \left(\{x\}-\frac{1}{2}\right) f_n^{'}(x)\,dx\right| &\le \frac{f_n(3n)+f_n(n+1)}{2} = \mathcal{O}(n^{-1}).
\end{aligned}$$

[1] See [1] p.117

This finally gives

$$S_n = \frac{1}{2}\left(\frac{\ln n}{2n} + \mathcal{O}(n^{-1})\right)^{-1} = \frac{n}{\ln n}\left(1 + \mathcal{O}(\ln^{-1} n)\right)^{-1} = \frac{n}{\ln n} + \mathcal{O}(n \ln^{-2} n)$$

and we immediately get what we wanted as $\pi(n) \sim \dfrac{n}{\ln n}$.

References

[1] H.S. Wilf *Mathematics for the physical sciences* , Dover Publications Inc., New York, 1962

A solution to the problem B -1091 of Volume 49 Number 3, August 2011 of The Fibonacci Quarterly

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Wednesday 9/11/2011

• **The Problem :** If $S_n = \left(\sum_{k=n}^{+\infty} \frac{1}{F_k}\right)^{-1}$, show that $S_n \sim \frac{a^{n-2}}{\sqrt{5}}$ as $n \to +\infty$.

• **Solution :** At first we observe that $\frac{1}{a-1} = a$ and $\left|\frac{b}{a}\right| = \frac{1}{a^2} < 1$. Now

$$\begin{aligned}
\sum_{k=n}^{+\infty} \frac{1}{F_k} &= \sqrt{5}\sum_{k=n}^{+\infty} a^{-k}\left(\frac{1}{1-\left(\frac{b}{a}\right)^k}\right) = \sqrt{5}\sum_{k=n}^{+\infty} a^{-k}\left(\frac{1}{1-(-a^{-2})^k}\right) \\
&= \sqrt{5}\sum_{k=n}^{+\infty} a^{-k}\left(1+\mathcal{O}(a^{-2k})\right) = \sqrt{5}\sum_{k=n}^{+\infty}\left(a^{-k}+\mathcal{O}(a^{-3k})\right) \\
&= \sqrt{5}\sum_{k=n}^{+\infty} a^{-k} + \mathcal{O}\left(\sum_{k=n}^{+\infty} a^{-3k}\right) = a^{-n+1}\frac{\sqrt{5}}{a-1} + \mathcal{O}\left(a^{-3k+3}\right) \\
&= \sqrt{5}a^{-n+2} + \mathcal{O}\left(a^{-3k+3}\right) \qquad (n \to +\infty), \qquad \text{so}
\end{aligned}$$

$$S_n = \left(\sqrt{5}a^{-n+2} + \mathcal{O}(a^{-3n+3})\right)^{-1} = \frac{a^{n-2}}{\sqrt{5}}\left(1+\mathcal{O}(a^{-2n+1})\right) = \frac{a^{n-2}}{\sqrt{5}} + \mathcal{O}(a^{-n-1}),$$

and from this we get directly the desired result.

A solution to the problem H-709 of Vol.49 No4 November's 2011 issue of The Fibonacci Quarterly

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May 21, 2012

The Problem : Proposed by Ovidiu Furdui, Campia Turzii, Romania.

a) *Let a be a positive real number. Calculate,*

$$\lim_{n\to+\infty} a^n(n-\zeta(2)-\zeta(3)-\ldots-\zeta(n)),$$

where ζ is the Riemann zeta function.

b) *Let a be a real number such that $|a|<2$. Prove that,*

$$\sum_{n=2}^{+\infty} a^n(n-\zeta(2)-\zeta(3)-\ldots-\zeta(n)) = a\left(\frac{\Psi(2-a)+\gamma}{1-a}-1\right),$$

where Ψ denotes the Digamma function.

Solution : We will use the well known identity

$$\Psi(x) = -\gamma + \sum_{m=1}^{+\infty}\left(\frac{1}{m}-\frac{1}{x+m-1}\right), \qquad x\in\mathbb{R}\setminus -\mathbb{N}\cup\{0\}. \tag{1}$$

Now

$$
\begin{aligned}
a^n\left(n-\sum_{k=2}^{n}\zeta(k)\right) &= a^n\left(1-\sum_{k=2}^{n}\sum_{m=2}^{+\infty}\frac{1}{m^k}\right)\\
{}^1 &= a^n\left(1-\sum_{m=2}^{+\infty}\sum_{k=2}^{n}\frac{1}{m^k}\right)\\
&= a^n\left(\sum_{m=2}^{+\infty}\frac{1}{m^2}\frac{1-m^{1-n}}{1-m^{-1}}\right)\\
&= a^n\left(1-\sum_{m=2}^{+\infty}\frac{1}{m^2-m}+\sum_{m=2}^{+\infty}\frac{1}{m^{n+1}-m^n}\right)\\
&= a^n\left(1-\sum_{m=2}^{+\infty}\left(\frac{1}{m-1}-\frac{1}{m}\right)+\sum_{m=2}^{+\infty}\frac{1}{m^{n+1}-m^n}\right)\\
&= \sum_{m=2}^{+\infty}\frac{a^n}{m^{n+1}-m^n}\\
&= \sum_{m=2}^{+\infty}\frac{1}{m-1}\left(\frac{a}{m}\right)^n,
\end{aligned}
$$

so,

a)

$$
a^n\left(n-\sum_{k=2}^{n}\zeta(k)\right)\begin{cases}
=\mathcal{O}\left(\left(1+\dfrac{a-2}{2}\right)^n\right)\to 0 & ,0<a<2\\[2ex]
=1+\mathcal{O}((2/3)^n)\to 1 & ,a=2\\[2ex]
>\left(1+\dfrac{a-2}{2}\right)^n\to+\infty & ,a>2
\end{cases}
$$

and

[1]Since the summands are positive

b)

$$
\begin{aligned}
\sum_{n=2}^{+\infty} a^n(n-\zeta(2)-\zeta(3)-\ldots-\zeta(n)) &= \sum_{n=2}^{+\infty}\sum_{m=2}^{+\infty}\frac{1}{m-1}\left(\frac{a}{m}\right)^n\\
&\overset{2}{=} \sum_{m=2}^{+\infty}\frac{1}{m-1}\sum_{n=2}^{+\infty}\left(\frac{a}{m}\right)^n\\
&= \sum_{m=2}^{+\infty}\frac{1}{m-1}\left(\frac{a}{m}\right)^2\left(\frac{1}{1-\frac{a}{m}}\right)\\
&= a^2\sum_{m=2}^{+\infty}\frac{1}{m(m-1)(m-a)}\\
&= a\sum_{m=1}^{+\infty}\left(\frac{1}{(m+1)}-\frac{1}{(1-a)(m+1-a)}+\frac{a}{(1-a)m}\right)\\
&= \frac{a}{1-a}\sum_{m=1}^{+\infty}\left(\frac{1}{m}-\frac{1}{m+1-a}\right)-a\sum_{m=1}^{+\infty}\left(\frac{1}{m}-\frac{1}{m+1}\right)\\
&\overset{(1)}{=} a\left(\frac{\Psi(2-a)+\gamma}{1-a}-1\right).
\end{aligned}
$$

[2]Since $|a|<2$, we have absolute convergence.

A solution to the problem 3616 of Volume 37 Number 2 issue of Crux Mathematicorum with Mathematical Mayhem

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Monday 14/11/2011

• **The Problem :** Proposed by Dinu Ovidiu Gabriel, Valcea, Romania.

Compute $\lim\limits_{n\to+\infty} n^{2k}\left(\dfrac{\tan^{-1} n^k}{n^k} - \dfrac{\tan^{-1}(n^k+1)}{n^k+1}\right)$, where $k\in\mathbb{R}$.

• **Solution :** At first we note that it is

$$\tan^{-1} x = x - \frac{x^3}{3} + \mathcal{O}(x^5) \qquad 0\le x\le 1,$$

and since $x\ge 1 \Rightarrow 0<\frac{1}{x}\le 1$, we get

$$\frac{\pi}{2} - \tan^{-1} x = \tan^{-1}\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \mathcal{O}(x^{-5}), \text{ so}$$

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + \mathcal{O}(x^{-5}) \qquad x\ge 1.$$

◇ If $k=0$ we clearly have $L = \dfrac{\pi}{4} - \dfrac{\tan^{-1} 2}{2}$.

◇ If $k<0$, it is $\dfrac{\tan^{-1} n^k}{n^k} = \dfrac{n^k + \mathcal{O}(n^{3k})}{n^k} = 1 + \mathcal{O}(n^{2k})$ so obviously $L=0$.

◇ If $k>0$, since $\dfrac{1}{1+n^k} = \dfrac{n^{-k}}{1+n^{-k}} = \dfrac{1}{n^k} - \dfrac{1}{n^{2k}} + \mathcal{O}(n^{-3k})$ we got

$$n^{2k}\left(\frac{\tan^{-1} n^k}{n^k} - \frac{\tan^{-1}(n^k+1)}{n^k+1}\right) =$$

$$n^{2k}\left(\frac{1}{n^k}\left(\frac{\pi}{2} - \frac{1}{n^k} + \mathcal{O}(n^{-3k})\right) - \frac{1}{n^k+1}\left(\frac{\pi}{2} - \frac{1}{n^k+1} + \mathcal{O}(n^{-3k})\right)\right) =$$

$$\frac{\pi}{2} + \mathcal{O}(n^k) \to \frac{\pi}{2}.$$

A solution to the problem 3618 of Volume 37 Number 2 issue of Crux Mathematicorum with Mathematical Mayhem

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Monday 14/11/2011

• **The Problem :** Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Let $a > 3$ be a real number. Find the value of $\displaystyle\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{n}{(n+m)^a}$.

• **Solution :** The summands being all positive we can sum by triangles :

$$\begin{aligned}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{n}{(n+m)^a} &= \sum_{n=2}^{+\infty}\frac{\sum_{k=1}^{n-1}k}{n^a}=\frac{1}{2}\sum_{n=2}^{+\infty}\frac{n(n-1)}{n^a}\\ &= \frac{1}{2}\left(\sum_{n=2}^{+\infty}\frac{1}{n^{a-2}}-\sum_{n=2}^{+\infty}\frac{1}{n^{a-1}}\right)\\ &= \frac{\zeta(a-2)-\zeta(a-1)}{2},\end{aligned}$$

where $\zeta(x)$ is the Riemann zeta function.

A solution to the problem 3624 of Volume 37 Number 2 issue of Crux Mathematicorum with Mathematical Mayhem

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April 9, 2013

The Problem : Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Calculate the sum

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^{n+1}}{n}\right).$$

Solution : For $x<1$ we set

$$f_m(x):=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\left(-\sum_{k=1}^{n}\frac{x^k}{k}\right),$$

so we search for $\lim\limits_{m\to+\infty} f_m(-1)$.

We have

$$\begin{aligned}
f_m'(x)&=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\left(-\sum_{k=1}^{n}\frac{x^k}{k}\right)'\\
&=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\left(-\sum_{k=0}^{n-1}x^k\right)\\
&=-\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\frac{1-x^n}{1-x}\\
&=\frac{1}{1-x}\sum_{n=1}^{m}\frac{(-1)^n}{n}+\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\cdot\frac{x^n}{1-x},
\end{aligned}$$

so we integrate from 0 to y, where $y<1$, to get

$$f_m(y)=\ln(1-y)\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}+\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n}\int_0^y\frac{x^n}{1-x}\,dx$$

and set $y = -1$ to get

$$\begin{aligned} f_m(-1) &= \ln 2 \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} + \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \int_0^{-1} \frac{x^n}{1-x}\,dx \\ &\overset{x=-t}{=\!=\!=} \ln 2 \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} + \sum_{n=1}^{m} \int_0^1 \frac{1}{n} \cdot \frac{t^n}{1+t}\,dt \\ &:= A_m + B_m. \end{aligned} \tag{1}$$

Now,

$$A_m \to \ln^2 2 \tag{2}$$

and from Monotone Convergence Theorem we have

$$B_m \to \int_0^1 \frac{1}{1+t} \sum_{n=1}^{+\infty} \frac{t^n}{n}\,dt = -\int_0^1 \frac{\ln(1-t)}{1+t}\,dt.$$

Furthermore,

$$\begin{aligned} -\int_0^1 \frac{\ln(1-t)}{1+t}\,dt &= \int_0^1 \int_{-1}^0 \frac{1}{1+t} \frac{t}{1+yt}\,dy\,dt \\ {}^1 &= \int_{-1}^0 \int_0^1 \frac{t}{(1+t)(1+yt)}\,dt\,dy \\ &= \int_{-1}^0 \int_0^1 \frac{1}{(y-1)(1+t)} - \frac{1}{(y-1)(1+yt)}\,dt\,dy \\ &= \int_{-1}^0 \frac{\ln 2}{y-1} - \frac{\ln(1+y)}{y(y-1)}\,dy \\ &= -\ln^2 2 + \int_{-1}^0 \frac{\ln(1+y)}{y} - \frac{\ln(1+y)}{y-1}\,dy \\ &\overset{y=-x}{=\!=\!=} -\ln^2 2 - \int_0^1 \frac{\ln(1-x)}{x}\,dx + \int_0^1 \frac{\ln(1-x)}{1+x}\,dx, \end{aligned}$$

so

[1]From Tonelli's theorem, since the integrand is non-negative.

$$
\begin{aligned}
\int_0^1 \frac{\ln(1-t)}{1+t}\,dt &= \frac{\ln^2 2}{2} + \frac{1}{2}\int_0^1 \frac{\ln(1-x)}{x}\,dx \\
&= \frac{\ln^2 2}{2} - \frac{1}{2}\int_0^1 \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n} \\
{}^{2} &= \frac{\ln^2 2}{2} - \frac{1}{2}\sum_{n=1}^{+\infty} \int_0^1 \frac{x^{n-1}}{n} \\
&= \frac{\ln^2 2}{2} - \frac{1}{2}\sum_{n=1}^{+\infty} \frac{1}{n^2} \\
&= \frac{\ln^2 2}{2} - \frac{\pi^2}{12}. \qquad (3)
\end{aligned}
$$

With (2) and (3), (1) will give

$$
\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}\right) = \frac{\ln^2 2}{2} + \frac{\pi^2}{12}.
$$

[2]From Monotone Convergence Theorem.

A solution to the problem 3637 of Volume 37 Number 3 issue of Crux Mathematicorum with Mathematical Mayhem

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Monday 14/11/2011

• **The Problem :** Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.

Let x be a real number with $|x| < 1$. Determine

$$\sum_{n=1}^{+\infty}(-1)^{n-1}n\left(\ln(1-x)+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right).$$

• **Solution :** For every $x \in (-1,1)$ and $m \in \mathbb{N}$ we have

$$\left(\sum_{n=1}^{m}(-1)^{n-1}n\left(\ln(1-x)+\sum_{k=1}^{n}\frac{x^k}{k}\right)\right)' =$$

$$\begin{aligned}\sum_{n=1}^{m}(-1)^{n-1}n\left(-\frac{1}{1-x}+\sum_{k=0}^{n-1}x^k\right) &= -\frac{x}{1-x}\sum_{n=1}^{m}(-1)^{n-1}nx^{n-1}\\ &= -\frac{x}{1-x}\left(\sum_{n=1}^{m}(-1)^{n-1}x^n\right)'\\ &= -\frac{x}{1-x}\left(\frac{x+(-x)^{m+1}}{1+x}\right)' =\end{aligned}$$

$$-\frac{x}{(1-x)(1+x)^2}+(-1)^m(m+1)\frac{x^{m+1}}{(1-x)(1+x)^2}+(-1)^m m\frac{x^{m+2}}{(1-x)(1+x)^2}, \qquad \text{so}$$

$$\sum_{n=1}^{m}(-1)^{n-1}n\left(\ln(1-x)+\sum_{k=1}^{n}\frac{x^k}{k}\right) =$$

$$-\int_0^x\frac{y}{(1-y)(1+y)^2}\,dy+(-1)^m\int_0^x\frac{(m+1)y^{m+1}}{(1-y)(1+y)^2}\,dy+(-1)^m\int_0^x\frac{my^{m+2}}{(1-y)(1+y)^2}\,dy$$

$$\xrightarrow{m\to+\infty}\frac{1}{2}\left(\frac{x}{x+1}-\tanh^{-1}x\right),$$

since the last two integrals go to zero as $m \to +\infty$ by the Dominated Convergence Theorem.

A solution to the problem 3512 p.46 of issue 36: No 1 FEBRUARY 2010 of Crux Mathematicorum with Mathematical Mayhem

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Tuesday 13/7/2010

Let $a \in \mathbb{R}$ and $\mathbb{R} \ni p \geq 1$.

We make use of the well known facts that:

$$\frac{x}{1+x} \leq \ln(1+x) \leq x, \qquad x > -1. \tag{1}$$

and that

If $f : [0,1] \to \mathbb{R}$ is a continuous function, then

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{k}{n}\Big) = \int_0^1 f(x)\,dx. \tag{2}$$

Let now $A_n := \prod_{k=1}^{n} \dfrac{n^p + (a-1)k^{p-1}}{n^p - k^{p-1}}$. Then

$$\ln A_n = \ln\Big(\prod_{k=1}^{n} \frac{n^p + (a-1)k^{p-1}}{n^p - k^{p-1}}\Big) = \sum_{k=1}^{n} \ln\Big(\frac{n^p + (a-1)k^{p-1}}{n^p - k^{p-1}}\Big) =$$

$$\sum_{k=1}^{n} \ln\Big(1 + \frac{ak^{p-1}}{n^p - k^{p-1}}\Big).$$

Using now (1) with $\dfrac{ak^{p-1}}{n^p - k^{p-1}}$ for x and considering that $\dfrac{ak^{p-1}}{n^p - k^{p-1}} \geq 0$ for $n \geq 2$, in the case $a \geq 0$, while $\dfrac{ak^{p-1}}{n^p - k^{p-1}} > -1$ for $n > (1-a)$, in the case $a < 0$, we have

$$\frac{\frac{ak^{p-1}}{n^p - k^{p-1}}}{1 + \frac{ak^{p-1}}{n^p - k^{p-1}}} \leq \ln\Big(1 + \frac{ak^{p-1}}{n^p - k^{p-1}}\Big) \leq \frac{ak^{p-1}}{n^p - k^{p-1}}. \tag{3}$$

for $k = 1, \cdots, n$.

It also is

$$1 + \frac{ak^{p-1}}{n^p - k^{p-1}} = 1 + \frac{a}{\frac{n^p}{k^{p-1}} - 1} \leq 1 + \frac{a}{n-1} = \frac{n+a-1}{n-1}$$

so (3) becomes

$$\frac{ak^{p-1}}{n^p - k^{p-1}} \cdot \frac{n-1}{n+a-1} \leq \ln\left(1 + \frac{ak^{p-1}}{n^p - k^{p-1}}\right) \leq \frac{ak^{p-1}}{n^p - k^{p-1}}$$

for $k = 1, \cdots, n$.

Adding up we have

$$a\frac{n-1}{n+a-1}\sum_{k=1}^{n}\frac{ak^{p-1}}{n^p - k^{p-1}} \leq \ln A_n \leq a\sum_{k=1}^{n}\frac{ak^{p-1}}{n^p - k^{p-1}}. \tag{4}$$

We proceed computing $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{n}\dfrac{ak^{p-1}}{n^p - k^{p-1}}$.

Writing $\sum\limits_{k=1}^{n}\dfrac{ak^{p-1}}{n^p - k^{p-1}} = \dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{\left(\frac{k}{n}\right)^{p-1}}{1 - \frac{k^{p-1}}{n^p}}$ and observing that

$$1 - \frac{1}{n} = 1 - \frac{n^{p-1}}{n^p} \leq 1 - \frac{k^{p-1}}{n^p} \leq 1, \qquad k = 1, \cdots, n$$

follows that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{p-1} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^{p-1}}{1 - \frac{k^{p-1}}{n^p}} \leq \frac{1}{1 - \frac{1}{n}} \cdot \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{p-1}.$$

Now we use (2) with $f(x) := x^{p-1}$ and the squeeze theorem to get that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left(\frac{k}{n}\right)^{p-1}}{1 - \frac{k^{p-1}}{n^p}} = \int_0^1 x^{p-1}\,dx = \frac{1}{p}.$$

Going back to (4) and using again the squeeze theorem one has

$$\lim_{n\to+\infty}\ln A_n = \frac{a}{p}.$$

Now we wright

$$\lim_{n\to+\infty}A_n = e^{\lim_{n\to+\infty}\ln A_n} = e^{\frac{a}{p}},$$

by the continuity of e^x, which concludes the proof.

A solution to the problem 3604 p.46 of issue 37: No 1 FEBRUARY 2011 of Crux Mathematicorum with Mathematical Mayhem

Anastasios Kotronis, Athens, Greece

Thursday 7/7/2011

We have

$$A := \frac{\int_0^1 (x^2-x-2)^n\,dx}{\int_0^1 (4x^2-2x-2)^n\,dx} = \frac{\int_0^1 (-x^2+x+2)^n\,dx}{\int_0^1 (-4x^2+2x+2)^n\,dx} := \frac{\int_0^1 f_n(x)\,dx}{\int_0^1 g_n(x)\,dx},$$

but we easily see that $f_n\left(\frac{1}{2}-x\right) = f_n\left(\frac{1}{2}+x\right)$ and $g_n\left(\frac{1}{4}-x\right) = g_n\left(\frac{1}{4}+x\right)$, so

$$A = \frac{2\int_0^{1/2} (-x^2+x+2)^n\,dx}{2\int_0^{1/4} (-4x^2+2x+2)^n\,dx + \int_{1/2}^1 (-4x^2+2x+2)^n\,dx}.$$

Now

$$\int_0^{1/4} (-4x^2 + 2x + 2)^n\,dx \overset{2x=y}{=} \frac{1}{2}\int_0^{1/2} (-y^2 + y + 2)^n\,dy \text{ and}$$

$$\int_{1/2}^{1} (-4x^2 + 2x + 2)^n\,dx \overset{2x=y+1}{=} \frac{1}{2}\int_0^{1} (-y^2 - y + 2)^n\,dy, \text{ so}$$

$$A = \frac{2\int_0^{1/2} (-x^2+x+2)^n\,dx}{\int_0^{1/2} (-x^2+x+2)^n\,dx + \frac{1}{2}\int_0^1 (-x^2-x+2)^n\,dx} := \frac{2a_n}{a_n + \frac{1}{2}b_n} \overset{a_n>0}{=} \frac{2}{1+\frac{b_n}{2a_n}}.$$

Since $2a_n = \int_0^1 (-x^2+x+2)^n\,dx > 2^n$, we have $0 < \frac{b_n}{2a_n} < \int_0^1 \left(\frac{-x^2-x+2}{2}\right)^n dx \to 0$ by the Dominated Convergence Theorem, for, $\left|\left(\frac{-x^2-x+2}{2}\right)^n\right| \leq 1\ \forall n \in \mathbb{N}\ \forall x \in [0,1]$ and $\left(\frac{-x^2-x+2}{2}\right)^n \xrightarrow{pw} \begin{cases} 0 & ,x \in (0,1] \\ 1 & ,x = 0 \end{cases}$. The required limit is hence 2.

Comments : **A)** Apart from the above "ad hoc" solution, there are powerful methods which can estimate the behavior of functions of the type $I_n = \int_a^b g(n,x)\,dx$ where $-\infty \leq a < b \leq +\infty$ as $n \to +\infty$ under specific conditions. A proposition know as "the Laplace's Method" is applicable here, namely :

Proposition 1. *Let* $\phi(t)$, $h(t) : [a,b] \to \mathbb{R}$, *where* $b \in \mathbb{R} \cup \{+\infty\}$, *functions for which the following hold*
i) $\phi(a) \neq 0$,
ii) $\phi(t)e^{xh(t)}$ *is absolutely integrable in* $[a,b]$ *for every* $x > 0$,
iii) $h(t)$ *has a unique maximum at* $t = a$ *and* $\sup\limits_{t\in I} h(t) < h(a)$, *where* I *any subinterval of* $[a,b]$ *with* $a \notin I$,
iv) $h''(t)$ *is continuous in a neighborhood of* a *and* $h'(a) = 0 \wedge h''(a) < 0$.
Then:

$$\lim_{x\to+\infty} x^{1/2}e^{-xh(a)}\int_a^b \phi(t)e^{xh(t)}\,dt = \phi(a)\sqrt{\frac{-\pi}{2h''(a)}}^{1}.$$

Proof: We assume that $\phi(a) > 0$. For the other case the proof is similar. Let $\varepsilon > 0$. From i), iv) and Taylor′s theorem, we find $\delta > 0$ with

$$t \in [a, a+\delta] \Rightarrow \begin{cases} 1)\ 0 < \phi(a) - \varepsilon \leq \phi(t) \leq \phi(t) + \varepsilon \\ 2)\ h''(a) - \varepsilon \leq h''(t) \leq h''(a) + \varepsilon < 0 \\ 3)\ h(t) = h(a) + \dfrac{1}{2}(t-a)^2 h''(\xi) \text{ forsome } \xi \in (a, a+\delta) \end{cases}.$$

From the above relations now, for $t \in [a, a+\delta]$, we have:

$$h''(a) - \varepsilon \leq h''(\xi) \leq h''(a) + \varepsilon \Rightarrow$$
$$\frac{1}{2}(t-a)^2(h''(a)-\varepsilon) \leq \frac{1}{2}(t-a)^2h''(\xi)\Big(= h(t) - h(a)\Big) \leq \frac{1}{2}(t-a)^2(h''(a)+\varepsilon) < 0 \Rightarrow$$
$$e^{\frac{x}{2}(t-a)^2\left(h''(a)-\varepsilon\right)+xh(a)} \leq e^{xh(t)} \leq e^{\frac{x}{2}(t-a)^2\left(h''(a)+\varepsilon\right)+xh(a)} \overset{\cdot\phi(t)}{\Rightarrow} {}^{2}$$
$$\phi(t)e^{\frac{x}{2}(t-a)^2\left(h''(a)-\varepsilon\right)+xh(a)} \leq \phi(t)e^{xh(t)} \leq \phi(t)e^{\frac{x}{2}(t-a)^2\left(h''(a)+\varepsilon\right)+xh(a)} \overset{1)}{\Rightarrow}$$
$$(\phi(a)-\varepsilon)e^{\frac{x}{2}(t-a)^2\left(h''(a)-\varepsilon\right)+xh(a)} \leq \phi(t)e^{xh(t)} \leq (\phi(a)+\varepsilon)e^{\frac{x}{2}(t-a)^2\left(h''(a)+\varepsilon\right)+xh(a)}.$$

Setting now $A := -(h''(a) - \varepsilon) > 0$, $B := -(h''(a) + \varepsilon) > 0$ and integrating from a to $a + \delta$, we get

$$(\phi(a)-\varepsilon)e^{xh(a)}\int_a^{a+\delta} e^{-\frac{x}{2}A(t-a)^2}\,dt \leq$$
$$\int_a^{a+\delta}\phi(t)e^{xh(t)}\,dt \leq$$
$$(\phi(a)+\varepsilon)e^{xh(a)}\int_a^{a+\delta} e^{-\frac{x}{2}B(t-a)^2}\,dt.$$

But

$$\int_a^{a+\delta} e^{-\frac{x}{2}A(t-a)^2}\,dt \overset{y=t-a}{=} \int_0^{\delta} e^{-\left(\sqrt{\frac{xA}{2}}y\right)^2}\,dy \overset{t=\sqrt{\frac{xA}{2}}y}{=}$$
$$\sqrt{\frac{2}{xA}}\left(\int_0^{+\infty} e^{-t^2}\,dt - \int_{\sqrt{\frac{xA}{2}}\delta}^{+\infty} e^{-t^2}\,dt\right) = \sqrt{\frac{\pi}{2xA}}\left(1 - \frac{2}{\sqrt{\pi}}\int_{\sqrt{\frac{xA}{2}}\delta}^{+\infty} e^{-t^2}\,dt\right) > 0$$

with $\int_{\sqrt{\frac{xA}{2}}\delta}^{+\infty} e^{-t^2}\,dt \overset{x\to+\infty}{\longrightarrow} 0$ and similarly for B instead of A. Hence

[1] This can equivalently be written $\int_a^b \phi(t)e^{xh(t)}\,dt \overset{x\to+\infty}{\sim} \phi(a)e^{xh(a)}\sqrt{\frac{-\pi}{2xh''(a)}}$, i.e. the function of x at the left, when $x \to +\infty$, behaves like the simpler function at the right.

[2] If we had assumed that $\phi(a) < 0$, we would have the reversed inequalities here

$$(\phi(a)-\varepsilon)e^{xh(a)}\sqrt{\frac{\pi}{2xA}}\cdot A(x)\leq \int_a^{a+\delta}\phi(t)e^{xh(t)}\,dt\leq (\phi(a)+\varepsilon)e^{xh(a)}\sqrt{\frac{\pi}{2xB}}\cdot B(x), \tag{1}$$

where $1-\dfrac{2}{\sqrt{\pi}}\displaystyle\int_{\sqrt{\frac{xA}{2}}\delta}^{+\infty}e^{-t^2}\,dt := A(x), B(x) := 1-\dfrac{2}{\sqrt{\pi}}\displaystyle\int_{\sqrt{\frac{xB}{2}}\delta}^{+\infty}e^{-t^2}\,dt \overset{x\to+\infty}{\longrightarrow} 1.$

To find bounds for the integral on the rest interval we write :

$$\left|\int_{a+\delta}^{b}\phi(t)e^{xh(t)}\,dt\right| = \int_{a+\delta}^{b}|\phi(t)|e^{h(t)+(x-1)h(t)}\,dt \overset{iii)}{\leq} \int_{a+\delta}^{b}|\phi(t)|e^{h(t)+(x-1)M}\,dt \overset{ii)}{\leq}$$

$$e^{(x-1)M}\int_a^b|\phi(t)|e^{h(t)}\,dt = Ke^{(x-1)M}, \tag{2}$$

for some $K>0$, where $M := \sup\limits_{x\in[a+\delta,b]} h(x) < h(a)$.

Now $((1)+(2))\cdot x^{1/2}e^{-xh(a)} \Rightarrow$

$$(\phi(a)-\varepsilon)\sqrt{\frac{\pi}{2A}}\cdot A(x)-\frac{Kx^{1/2}}{e^M}e^{x(M-h(a))}\leq x^{1/2}e^{-xh(a)}\int_a^b\phi(t)e^{xh(t)}\,dt\leq (\phi(a)+\varepsilon)\sqrt{\frac{\pi}{2B}}\cdot B(x)+\frac{Kx^{1/2}}{e^M}e^{x(M-h(a))},$$

from were we have that

$$(\phi(a)-\varepsilon)\sqrt{\frac{-\pi}{2(h''(a)-\varepsilon)}}\leq {}^{3}\liminf,\limsup x^{1/2}e^{-xh(a)}\int_a^b\phi(t)e^{xh(t)}\,dt\leq (\phi(a)+\varepsilon)\sqrt{\frac{-\pi}{2(h''(a)+\varepsilon)}}.$$

Now letting ε go to 0, we get

$$\lim_{x\to+\infty}x^{1/2}e^{-xh(a)}\int_a^b\phi(t)e^{xh(t)}\,dt = \phi(a)\sqrt{\frac{-\pi}{2h''(a)}},\text{ as we wanted.}$$

B) 1) In the case that h has its maximum at an interior point x_0 of $[a,b]$, then, integrating seperately at the subintervals $[a,x_0]$ and $[x_0,b]$, and applying the proposition for each of them, we get

[3] We assume that x goes to $+\infty$ through an arbitrary sequence x_n and we do not show the index throughout the proof. So $\liminf, \limsup$, refer to n

$$\int_a^b \phi(t)e^{xh(t)}\,dt \overset{x\to+\infty}{\sim} 2\phi(x_0)e^{xh(x_0)}\sqrt{\frac{-\pi}{2xh''(x_0)}} = \phi(x_0)e^{xh(x_0)}\sqrt{\frac{-2\pi}{xh''(x_0)}}.$$

2) If the rest of the conditions of the proposition hold, and h has its maximum at the endpoint a with $h^{'}(a) \neq 0$, then approximating $h(t)$ with $h(a) + h^{'}(a)(t-a)$ at the neighborhood of a we end up with

$$\int_a^b \phi(t)e^{xh(t)}\,dt \overset{x\to+\infty}{\sim} -\frac{\phi(a)e^{xh(a)}}{xh'(a)}.$$

3) If the rest of the conditions of the proposition hold, and h has its maximum at the endpoint $b < +\infty$ with $h^{'}(b) \neq 0$, then similarly with 2), we get

$$\int_a^b \phi(t)e^{xh(t)}\,dt \overset{x\to+\infty}{\sim} \frac{\phi(b)e^{xh(b)}}{xh'(b)}.$$

4) If in the general case we have global maximum at the point $c \in (a, b)$, with $h^{'}(c) = h^{''}(c) = \ldots = h^{(m-1)}(c) = 0$ and $h^{(m)}(c) = 0$, then

$$\int_a^b \phi(t)e^{xh(t)}\,dt \overset{x\to+\infty}{\sim} \frac{2\Gamma(1/m)\phi(c)e^{x\phi(c)}}{m}\left(\frac{m!}{-xh^{(m)}(c)}\right)^{1/m}.$$

If specificaly we have $c = a$, we have the above relation multiplied by $-1/2$ at the right, and if $c = b < +\infty$, we have the above relation multiplied by $1/2$ at the right.

C) For the proposed problem, using the proposition, writing $\int_0^1 (-x^2+x+2)^n\,dx = \int_0^1 \exp\Big(n\ln(-x^2+x+2)\Big)\,dx$ and $\int_0^1 (-4x^2+2x+2)^n\,dx = \int_0^1 \exp\Big(n\ln(-4x^2+2x+2)\Big)\,dx$, we have the conditions fulfilled so immediatelly

$$\frac{\int_0^1(-x^2+x+2)^n\,dx}{\int_0^1(-4x^2+2x+2)^n\,dx} \overset{n\to+\infty}{\sim} \frac{e^{9n/4}\sqrt{\frac{\pi}{16n/9}}}{e^{9n/4}\sqrt{\frac{\pi}{64n/9}}} = 2$$

and the result follows.

D) The above proof was a "details added version" of the one presented on the first book of the list below. A proof which gives a better estimation of the function defined by the parametric integral can be found on the second book of the list below.

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Problem of the Week

Department of Mathematics, Purdue University

Fall 2012 - Problem No. 4

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September 20, 2012

The Problem :

(a) Prove that if f is continuous on $[0,1]$ and differentiable on $(0,1)$ and satisfies $|f'(x)| \le M$ for some positive number M then

$$\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\right| \le \frac{M}{n}.$$

(b) (optional; the problem will be counted as solved if part (a) is solved) Show that the $\frac{M}{n}$ of part (a) can be improved to $\frac{M}{2n}$.

Solution :

(a)

$$\begin{aligned}
\left|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\right| &= \left|\sum_{k=0}^{n-1}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) - f\left(\frac{k}{n}\right)\,dx\right)\right| \\
&\le \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(x-\frac{k}{n}\right)\frac{f(x)-f\left(\frac{k}{n}\right)}{x-\frac{k}{n}}\,dx\right| \\
&\overset{1}{=} \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(x-\frac{k}{n}\right)f'(\xi)\,dx\right| \\
&\overset{2}{\le} M\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\frac{k+1}{n}-\frac{k}{n}\right)\,dx \\
&= \frac{M}{n}.
\end{aligned}$$

[1]From the Mean Value Theorem, for some $\xi \in \left(\frac{k}{n}, \frac{k+1}{n}\right) \subseteq (0,1)$.

[2]Since $|f'(x)| \le M$ for $x \in (0,1)$.

(b) As in (a),

$$\begin{aligned}\left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{k=0}^{n-1}f\left(\frac{k}{n}\right)\right| &\leq \sum_{k=0}^{n-1}\left|\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(x-\frac{k}{n}\right)f^{'}(\xi)\,dx\right|\\ &\leq M\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}x-\frac{k}{n}\,dx\\ &=\frac{M}{2n}.\end{aligned}$$

Problem of the Week

Department of Mathematics, Purdue University

Fall 2012 - Problem No. 13

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November 20, 2012

The Problem : What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \leq e \leq \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n?

Solution : The problem is equivalent to finding the maximum value of a and the minimum value of b for which

$$\begin{cases} b \geq \dfrac{1}{\ln(1+1/n)} - n \\ \qquad \text{and} \\ a \leq \dfrac{1}{\ln(1+1/n)} - n \end{cases} \quad \forall n \in \mathbb{N}.$$

Setting $f(x) := \dfrac{1}{\ln(1+x)} - \dfrac{1}{x}, \qquad x \in (0,1]$, we have that

$$f'(x) = \left(\ln(1+x) - \frac{x}{\sqrt{1+x}}\right) \frac{\sqrt{1+x}\ln(1+x) + x}{x^2\sqrt{1+x}\ln^2(1+x)}, \qquad x \in (0,1].$$

But $\dfrac{d}{dx}\left(\ln(1+x) - \dfrac{x}{\sqrt{1+x}}\right) = \dfrac{2\sqrt{1+x} - (x+2)}{2(1+x)^{3/2}} < 0$ for $x \in (0,1]$ and since $\lim\limits_{x\to 0^+} \ln(1+x) - \dfrac{x}{\sqrt{1+x}} = 0$ we get that $f'(x) < 0$ on $(0,1]$ so f is strictly decreasing on $(0,1]$.
Furthermore, $f(1) = \dfrac{1}{\ln 2} - 1$ and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{1}{x - x^2/2 + \mathcal{O}(x^2)} - \dfrac{1}{x} = \dfrac{1}{2}$ so the desired values are $a = \dfrac{1}{\ln 2} - 1$ and $b = \dfrac{1}{2}$.

A solution to the problem 188 of Vol.14 Núm.4 2011 of La Gaceta De la RSME

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November 28, 2012

The Problem : Proposed by Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Rumanía.

Let $\{x_n\}_{n\geq 1}$ *be the sequence defined by* $x_1 = 2$ *and* $x_{n+1} = \dfrac{1}{1+x_n}$ *for* $n \geq 1$*. Evaluate* $\displaystyle\prod_{n=1}^{+\infty} x_n$.

Solution : We set $x_n = \dfrac{a_n}{b_n}$, so from the given relation we have $\dfrac{a_{n+1}}{b_{n+1}} = \dfrac{2}{1+\frac{a_n}{b_n}} = \dfrac{2b_n}{a_n+b_n}$ and hence we can write

$$\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} := A^n \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{1}$$

But A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, so for

$$P =: \begin{bmatrix} 2 & 2 \\ \lambda_1 - 0 & \lambda_2 - 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \quad \text{with} \quad P^{-1} = \begin{bmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\ \dfrac{1}{6} & \dfrac{1}{3} \end{bmatrix}$$

we get

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \text{and hence} \quad A^n = P \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix} P^{-1} = \begin{bmatrix} \dfrac{2(-1)^n + 2^n}{3} & \dfrac{2(-1)^{n+1} + 2^{n+1}}{3} \\ \dfrac{(-1)^{n+1} + 2^n}{3} & \dfrac{(-1)^{n+2} + 2^{n+1}}{3} \end{bmatrix},$$

so (1) will give

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} \dfrac{2(-1)^{n-1} + 2^{n+1}}{3} \\ \dfrac{(-1)^n + 2^{n+1}}{3} \end{bmatrix}.$$

Finally,

$$x_n = \frac{2\left((-1)^{n-1} + 2^n\right)}{(-1)^n + 2^{n+1}} \Rightarrow \prod_{n=1}^{N} x_n = \frac{2^{N-1}3}{(-1)^{N-1} + 2^N} = \frac{3}{\left(-\frac{1}{2}\right)^{N-1} + 2} \to \frac{3}{2}.$$

A solution to the problem 190 of Vol.14 Núm.4 2011 of La Gaceta De la RSME

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May 20, 2012

The Problem : Proposed by Pablo Fernández Refolio (estudiante), Universidad Autónoma de Madrid, Madrid.

Denoting by A *the Glaisher - Kinkelin constant, which is defined from*

$$A = \lim_{n\to+\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.2824\ldots,$$

prove that

$$\prod_{n=1}^{+\infty} \left(e^{-2n} \left(1+\frac{1}{n}\right)^{2n^2+n-1/6} \right) = \frac{\sqrt{2\pi}}{A^4}.$$

Solution : It suffices to show that

$$C_n := n^{-2n^2-2n-1/3} e^{n^2} \prod_{k=1}^{n} k^{4k} \prod_{k=1}^{n} e^{-2k} \left(1+\frac{1}{k}\right)^{2k^2+k-1/6} \to \sqrt{2\pi}.$$

But

$$\begin{aligned}
C_n &= n^{-2n^2-2n-1/3} e^{n^2} \prod_{k=1}^{n} (k+1) e^{-2k} \frac{(k+1)^{2(k+1)^2-3(k+1)-1/6}}{k^{2k^2-3k-1/6}} \\
&= \frac{(n+1)^{2(n+1)^2-3(n+1)-1/6}}{n^{2n^2+2n+1/3}} e^{n^2} \prod_{k=1}^{n} (k+1) e^{-2k} \\
&= \frac{(n+1)^{2n^2+n-7/6}}{n^{2n^2+2n+1/3}} e^{n^2} \frac{(n+1)!}{e^{n(n+1)}} \\
&= \left(1+\frac{1}{n}\right)^{2n^2+n-1/6} \frac{n!}{n^{n+1/2} e^{n}}
\end{aligned}$$

and from stirling's formula it is $n! = \dfrac{\sqrt{2\pi}n^{n+1/2}}{e^n}\left(1+\mathcal{O}(n^{-1})\right)$, so

$$\begin{aligned}
C_n &= \sqrt{2\pi}e^{-2n}\left(1+\frac{1}{n}\right)^{2n^2+n-1/6}\left(1+\mathcal{O}(n^{-1})\right) \\
&= \sqrt{2\pi}\exp\left(-2n+\left(2n^2+n-1/6\right)\ln\left(1+\frac{1}{n}\right)\right)\left(1+\mathcal{O}(n^{-1})\right) \\
&= \sqrt{2\pi}\exp\left(-2n+\left(2n^2+n-1/6\right)\left(\frac{1}{n}-\frac{1}{2n^2}+\mathcal{O}(n^{-3})\right)\right)\left(1+\mathcal{O}(n^{-1})\right) \\
&= \sqrt{2\pi}e^{\mathcal{O}(n^{-1})}\left(1+\mathcal{O}(n^{-1})\right) \to \sqrt{2\pi}.
\end{aligned}$$

A solution to the problem 1260 of Spring's 2012 issue of Pi Mu Epsilon journal

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April 25, 2012

The Problem : Proposed by Paul Bruckman, Nanaimo, British Columbia.

Prove the following identity, valid for $n = 0, 1, 2, \dots$: .

$$\sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{n-k}{2k} 4^k 3^{n-3k} = \frac{1}{9}\left(4^{n+1} + 6n + 5\right).$$

Solution : We use that

$$\binom{n}{m} = 0 \qquad 0 \le n < m, \quad n, m \in \mathbb{N} \tag{1}$$

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n\ge 0} \binom{n}{k} x^n \qquad k \in \mathbb{N} \cup \{0\} \tag{2}$$

$$\frac{1}{(1-x)^2} = \sum_{n\ge 0} (n+1)x^n \tag{3}$$

and compute the generating function of the sequence $\sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{n-k}{2k} 4^k 3^{n-3k}$.

We have

$$
\begin{aligned}
\sum_{n=0}^{+\infty}\sum_{k=0}^{\left[\frac{n}{3}\right]}\binom{n-k}{2k}4^k3^{n-3k}x^n &= \sum_{k\geq 0}\sum_{n\geq 3k}\binom{n-k}{2k}4^k3^{3-nk}x^n \\
&= \sum_{k\geq 0}4^k3^{-2k}x^k\sum_{n\geq 3k}\binom{n-k}{2k}(3x)^{n-k} \\
&\overset{n-k=m}{=\!=\!=} \sum_{k\geq 0}\left(\frac{2}{3}\right)^{2k}x^k\sum_{m\geq 2k}\binom{m}{2k}(3x)^m \\
&\overset{(1)}{=} \sum_{k\geq 0}\left(\frac{2}{3}\right)^{2k}x^k\sum_{m\geq 0}\binom{m}{2k}(3x)^m \\
&\overset{(2)}{=} \sum_{k\geq 0}\left(\frac{2}{3}\right)^{2k}x^k\frac{(3x)^{2k}}{(1-3x)^{2k+1}} \\
&= \frac{1}{1-3x}\sum_{k\geq 0}\left(\frac{4x^3}{(1-3x)^2}\right)^k \\
&= \frac{1}{1-3x}\cdot\frac{1}{1-\frac{4x^3}{(1-3x)^2}} \\
&= \frac{3x-1}{(x-1)^2(4x-1)} \\
&= \frac{4}{9(1-4x)}-\frac{1}{9(1-x)}+\frac{2}{3(1-x)^2} \\
&\overset{(3)}{=} \frac{4}{9}\sum_{n=0}^{+\infty}(4x)^n-\frac{1}{9}\sum_{n=0}^{+\infty}x^n+\frac{2}{3}\sum_{n=0}^{+\infty}(n+1)x^n \\
&= \sum_{n=0}^{+\infty}\frac{4^{n+1}+6n+5}{9}x^n.
\end{aligned}
$$

Equating coefficients we get the desired result.

Problem of the Week

Department of Mathematics, Purdue University

Spring 2012 - Problem No. 2

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January 24, 2012

The Problem : *Find* $\lim_{n\to+\infty} \dfrac{1^1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$.

Solution :

We have

$$\begin{aligned}\lim_{n\to+\infty} \frac{\sum_{k=1}^{n+1} k^k - \sum_{k=1}^{n} k^k}{(n+1)^{n+1}-n^n} &= \lim_{n\to+\infty} \frac{(n+1)^{n+1}}{(n+1)^{n+1}-n^n}\\ &= \lim_{n\to+\infty} \frac{1}{1-\frac{\left(1+\frac{1}{n}\right)^{-n}}{n+1}}\\ &= \frac{1}{1-0\cdot e^{-1}}\\ &= 1\end{aligned}$$

and since n^n increases to $+\infty$, by Cesàro - Stolz theorem it is

$$\lim_{n\to+\infty} \frac{1^1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n} = \lim_{n\to+\infty} \frac{\sum_{k=1}^{n+1} k^k - \sum_{k=1}^{n} k^k}{(n+1)^{n+1}-n^n} = 1.$$

Problem of the Week

Department of Mathematics, Purdue University

Spring 2013 - Problem No. 5

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February 6, 2013

The Problem. *Find* $\lim\limits_{n\to+\infty}\int_0^{+\infty}\frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx$.

Solution 1: We have

$$\int_0^{+\infty}\frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx \overset{y=nx}{=} \int_0^{\infty}\frac{e^{-y/n}}{y^2+1}\cos\left(\frac{y}{n}\right)dy$$

but $\left|\frac{e^{-y/n}}{y^2+1}\cos\left(\frac{y}{n}\right)\right| \leq \frac{1}{y^2+1}, \qquad y \geq 0$ with $\int_0^{+\infty}\frac{1}{y^2+1}\,dy = \frac{\pi}{2}$ so, since

$$\frac{e^{-y/n}}{y^2+1}\cos\left(\frac{y}{n}\right) \to \frac{1}{y^2+1}, \qquad y \geq 0,$$

from Dominated Convergence Theorem we get

$$\int_0^{+\infty}\frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx \to \int_0^{+\infty}\frac{1}{y^2+1}\,dy = \frac{\pi}{2}.$$

Solution 2: We have

$$\begin{aligned}\int_0^{+\infty}\frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx &= \left(\cos x e^{-x}\tan^{-1}(nx)\Big|_0^{+\infty} + \int_0^{+\infty}\tan^{-1}(nx)e^{-x}(\sin x+\cos x)\,dx\right)\\ &= \int_0^{+\infty}\tan^{-1}(nx)e^{-x}(\sin x+\cos x)\,dx\end{aligned}$$

but $\left|\tan^{-1}(nx)e^{-x}(\sin x + \cos x)\right| \leq \pi e^{-x}, \qquad x \geq 0$ with $\displaystyle\int_0^{+\infty} e^{-x}\,dx = 1$ so, since

$$\tan^{-1}(nx)e^{-x}(\sin x + \cos x) \to \begin{cases} \dfrac{\pi}{2}e^{-x}(\sin x + \cos x) & , x \geq 0 \\ 0 & , x = 0 \end{cases},$$

from Dominated Convergence Theorem we get

$$\int_0^{+\infty} \frac{e^{-x}\cos x}{\frac{1}{n} + nx^2}\,dx \to \frac{\pi}{2}\int_0^{+\infty} e^{-x}(\sin x + \cos x)\,dx = \frac{\pi}{2}.$$

Comment: This problem appears at *Problems and Solutions in Mathematics* by *Ta-tsien Li, Chen Ji-Xiu*, Second edition 2011 p.422.

Problem of the Week

Department of Mathematics, Purdue University

Spring 2013 - Problem No. 7

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February 27, 2013

The Problem. *Find the radius of convergence of the MacLaurin expansion of* $\displaystyle\int_0^{+\infty}\frac{1}{e^t+xt}\,dt$.

Solution : Let b be a real number with $0<b<1$ and $x\in\mathbb{R}$ such that $\left|\frac{xt}{e^t}\right|\leq b$ for all $t\geq 0$, which gives that $|x|\leq be$. With the above conditions we can write:

$$\begin{aligned}\int_0^{+\infty}\frac{1}{e^t+xt}\,dt &= \int_0^{+\infty}\frac{e^{-t}}{1+\frac{xt}{e^t}}\,dt=\int_0^{+\infty}\sum_{n\geq 1}(-xt)^{n-1}e^{-nt}\,dt\\ &\stackrel{*}{=}\sum_{n\geq 1}(-x)^{n-1}\int_0^{+\infty}t^{n-1}e^{-nt}\,dt\stackrel{kt=z}{=}\sum_{n\geq 1}\frac{(-x)^{n-1}}{n^{n-1}}\int_0^{+\infty}e^{-z}z^{n-1}\,dz\\ &=\sum_{n\geq 0}(-1)^n\frac{n!}{(n+1)^n}x^n\end{aligned}$$

with the change of integration and summation order in $*$ being justified from Fubini's theorem (`http://en.wikipedia.org/wiki/Fubini%27s_theorem#Theorem_statement`), since $|(-xt)^{n-1}e^{-nt}|\leq b^{n-1}e^{-t}$ and $\displaystyle\int_0^{+\infty}\sum_{n\geq 0}b^ne^{-t}\,dt$ converges.

Since b was arbitrary we get that the radius of convergence of the MacLaurin expansion of $\displaystyle\int_0^{+\infty}\frac{1}{e^t+xt}\,dt$ is e and, furthermore that it's MacLaurin expansion is $\displaystyle\sum_{n\geq 0}(-1)^n\frac{n!}{(n+1)^n}x^n$.

Comment : It can, furthermore, be shown that near the singularity $x=-e$, we have

$$\int_0^{+\infty}\frac{1}{e^t+xt}\,dt\sim\pi\sqrt{\frac{2}{e}}(x+e)^{-1/2},$$

in the sense that the quotient of the two quantities converges to 1 as $x\to -e^+$.[1]

[1] See `http://www.artofproblemsolving.com/Forum/viewtopic.php?f=67&t=521949`.

A solution to the problem #679 of Spring 2011 issue of The Pentagon journal

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Saturday 26/11/2011

• **The Problem :** Proposed by Hongbiao Zeng, Fort Hays State University, Hays, KS.

> Suppose that $f(x)$ is continuous and bounded on $(0, +\infty)$ and the sequence $\{f(n)\}_{n=1}^{+\infty}$ doesn't converge. Show that for any positive constant M, there exists an $x_0 > M$ such that $f(x_0 + 1) > f(x_0)$.

• **Solution :** Suppose that there exists a positive constant M such that for every $x > M$ it is $f(x) \geq f(x+1)$. Applying this for $x = n_0 := [M] + 1, n_0 + 2, \ldots 3$, we get

$$f(n_0) \geq f(n_0 + 1) \geq f(n_0 + 2) \geq \ldots,$$

which means that $\{f(n)\}_{n=1}^{+\infty}$ is finaly decreasing and since $f(x)$ is bounded, $\{f(n)\}_{n=1}^{+\infty}$ is convergent, which is a contradiction. The assumption of the continuity is not necessary.

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Iterated sin sequence

Let $x_0 \in (0, \pi)$ fixed. For $n \in \mathbb{N}$ we set $x_n = \sin x_{n-1}$. Show that

$$x_n = \frac{3^{1/2}}{n^{1/2}} - \frac{3^{3/2} \ln n}{10 n^{3/2}} + \mathcal{O}(n^{-3/2}).$$

The above exercise generalizes the well known one, found in the most analysis problem books, which asks $\lim\limits_{n \to +\infty} \sqrt{n} x_n$ to be found.

The presented solution has the advantage that can be applied to many cases of an $x_n = f(x_{n-1})$ iteration in an almost identical way, it provides more information than just finding the coefficients in the asymptotic expansion of x_n and that the coefficients themselves occur in a more "natural" way than from "ad' hoc" successive applications of Cesàro Stolz theorem.

solution

We directly see by induction that $x_n \in (0, \pi)$ so $x_{n+1} = \sin x_n < x_n$, hence x_n is strictly decreasing and converges to $\ell = 0$, since $\ell = \sin \ell$ holds.
From the taylor expansion of $\sin$ we have

$$x_{n+1} = x_n - \frac{x_n^3}{6} + \frac{x_n^5}{120} + \mathcal{O}(x_n^7), \qquad (n \to +\infty)$$

and we use this and the taylor expansion of $(1+x)^a$ to find an $a \in \mathbb{R}$ for which $x_{n+1}^a - x_n^a \to k \in \mathbb{R}^*$.

$$\begin{aligned}
x_{n+1}^a - x_n^a &= \left(x_n - \frac{x_n^3}{6} + \frac{x_n^5}{120} + \mathcal{O}(x_n^7) \right)^a - x_n^a \\
&= x_n^a \left(\left(1 - \frac{x_n^2}{6} + \frac{x_n^4}{120} + \mathcal{O}(x_n^6) \right)^a - 1 \right) \\
&= x_n^a \left(-\frac{a x_n^2}{6} + \frac{a x_n^4}{120} + \frac{a(a-1)}{2} \left(-\frac{x_n^2}{6} + \frac{x_n^4}{120} + \mathcal{O}(x_n^6) \right)^2 + \mathcal{O}(x_n^6) \right) \\
&= -\frac{a}{6} x_n^{a+2} + \frac{3a + 5a(a-1)}{360} x_n^{a+4} + \mathcal{O}(x_n^{a+6}),
\end{aligned}$$

so for $a = -2$ we get

$$x_{n+1}^{-2} - x_n^{-2} = \frac{1}{3} + \frac{x_n^2}{15} + \mathcal{O}(x_n^4) \tag{1}$$

Since $x_{n+1}^{-2} - x_n^{-2} \to \frac{1}{3}$, for n big we have $x_{n+1}^{-2} - x_n^{-2} > \frac{1}{6}$, so summing this for n consecutive terms we get $x_n^{-2} > \frac{n}{6}$ and hence

$$x_n^2 = \mathcal{O}(n^{-1}).$$

With this estimate, (1) becomes $x_{n+1}^{-2} - x_n^{-2} = \frac{1}{3} + \mathcal{O}(n^{-1})$ so summing up again we get

$$x_n^{-2} = \frac{n}{3} + \mathcal{O}(\ln n).$$

Plugging again in (1) we get

$$\begin{aligned}
x_{n+1}^{-2} - x_n^{-2} &= \frac{1}{3} + \frac{1}{5n + \mathcal{O}(\ln n)} + \mathcal{O}(n^{-2}) \\
&= \frac{1}{3} + \frac{1}{5n}\left(\frac{1}{1 + \mathcal{O}(n^{-1}\ln n)}\right) + \mathcal{O}(n^{-2}) \\
&= \frac{1}{3} + \frac{1}{5n}\left(1 + \mathcal{O}(n^{-1}\ln n)\right) + \mathcal{O}(n^{-2}) \\
&= \frac{1}{3} + \frac{1}{5n} + \mathcal{O}(n^{-2}\ln n).
\end{aligned}$$

so summing once more, since $\sum_{n=1}^{+\infty} \frac{\ln n}{n^2}$ is convergent, we get

$$x_n^{-2} = \frac{n}{3} + \frac{\ln n}{5} + \mathcal{O}(1).$$

Finally

$$\begin{aligned}
x_n &= \left(\frac{n}{3} + \frac{\ln n}{5} + \mathcal{O}(1)\right)^{-1/2} \\
&= \frac{3^{1/2}}{n^{1/2}}\left(1 + \frac{3\ln n}{5n} + \mathcal{O}(n^{-1})\right)^{-1/2} \\
&= \frac{3^{1/2}}{n^{1/2}} - \frac{3^{3/2}\ln n}{10n^{3/2}} + \mathcal{O}(n^{-3/2}).
\end{aligned}$$

A solution to the problem U220 of issue 1, 2012 of Mathematical Reflections

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April 20, 2012

The Problem : Proposed by Cezar Lupu, University of Pittsburgh, USA and Moubinool Omarjee, Lycee Jean Murcat, Paris, France.

Evaluate

$$\lim_{n\to+\infty}\left((n+1)\sqrt[n+1]{\Gamma\left(\frac{1}{n+1}\right)}-n\sqrt[n]{\Gamma\left(\frac{1}{n}\right)}\right),$$

where Γ *denotes the classical Gamma function.*

Solution :

It is well known that

$$\Gamma(x)=\frac{1}{x}-\gamma+\mathcal{O}(x)\qquad x\to 0. \tag{1}$$

With this we get

$$\begin{aligned}
\sqrt[n+1]{\Gamma\left(\frac{1}{n+1}\right)} &= \exp\left(\frac{1}{n+1}\ln\Gamma\left(\frac{1}{n+1}\right)\right)\\
&\overset{(1)}{=} \exp\left(\frac{1}{n+1}\ln\left(n+1-\gamma+\mathcal{O}(n^{-1})\right)\right)\\
&= \exp\left(\left(\frac{1}{n}+\mathcal{O}(n^{-1})\right)\left(\ln n+\ln\left(1+\frac{1-\gamma}{n}+\mathcal{O}(n^{-2})\right)\right)\right)\\
&= \exp\left(\left(\frac{1}{n}+\mathcal{O}(n^{-1})\right)\left(\ln n+\frac{1-\gamma}{n}+\mathcal{O}(n^{-2})\right)\right)\\
&= \exp\left(\frac{\ln n}{n}+\mathcal{O}(n^{-2}\ln n)\right)\\
&= 1+\frac{\ln n}{n}+\mathcal{O}(n^{-2}\ln^2 n)
\end{aligned} \tag{2}$$

and similarly

$$\sqrt[n]{\Gamma\left(\frac{1}{n}\right)} = 1 + \frac{\ln n}{n} + \mathcal{O}(n^{-2}\ln^2 n). \tag{3}$$

Now (2) and (3) give

$$(n+1)\sqrt[n+1]{\Gamma\left(\frac{1}{n+1}\right)} - n\sqrt[n]{\Gamma\left(\frac{1}{n}\right)} = 1 + \mathcal{O}(n^{-1}\ln^2 n) \xrightarrow{n\to+\infty} 1.$$

A solution to the problem J256 of issue 1 2013 of Mathematical Reflections

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January 28, 2013

The Problem. *Proposed by Titu Andreescu, University of Texas at Dallas, USA*

Evaluate

$$1^2 2! + 2^2 3! + \cdots + n^2(n+1)!.$$

Solution : For $n \geq 2$ we have

$$\begin{aligned}
1^2 2! + 2^2 3! + \cdots + n^2(n+1)! &= 2 + \sum_{k=2}^{n} k^2(k+1)! \\
&= 2 + \sum_{k=2}^{n} (k+1)!\,((k-1)(k+2) - (k-2)) \\
&= 2 + \sum_{k=2}^{n} ((k+2)!(k-1) - (k+1)!(k-2)) \\
&= 2 + (n+2)!(n-1).
\end{aligned}$$

Since the last expression, for $n = 1$, equals $1^2 2!$ we get that

$$1^2 2! + 2^2 3! + \cdots + n^2(n+1)! = 2 + (n+2)!(n-1), \qquad \text{for } n \text{ a positive integer.}$$

A solution to the problem U253 of issue 1 2013 of Mathematical Reflections

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June 3, 2013

The Problem. *Proposed by Titu Andreescu, University of Texas at Dallas, USA*

Evaluate

$$\sum_{n\geq 2}\frac{3n^2+1}{(n^3-n)^3}.$$

Solution : Decomposing into partial fractions we get

$$\frac{3n^2+1}{(n^3-n)^3}=\frac{1}{2}\left(\frac{1}{(n+1)^3}-\frac{2}{n^3}+\frac{1}{(n-1)^3}\right)+\frac{3}{2}\left(\frac{1}{(n+1)^2}-\frac{1}{(n-1)^2}\right)+3\left(\frac{1}{n+1}-\frac{2}{n}+\frac{1}{n-1}\right),$$

thus

$$\sum_{n\geq 2}\frac{3n^2+1}{(n^3-n)^3}=\lim_{N\to+\infty}\sum_{n=2}^{N}\frac{3n^2+1}{(n^3-n)^3}=$$

$$\lim_{N\to+\infty}\left(\frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{(n+1)^3}-\frac{2}{n^3}+\frac{1}{(n-1)^3}\right)+\frac{3}{2}\sum_{n=2}^{N}\left(\frac{1}{(n+1)^2}-\frac{1}{(n-1)^2}\right)+3\sum_{n=2}^{N}\left(\frac{1}{n+1}-\frac{2}{n}+\frac{1}{n-1}\right)\right)=$$

$$\lim_{N\to+\infty}\frac{1}{2}\left(\frac{1}{N^3}+\frac{1}{(N+1)^3}-\frac{2}{N^3}-\frac{2}{2^3}+1+\frac{1}{2^3}\right)+\lim_{N\to+\infty}\frac{3}{2}\left(\frac{1}{N^2}+\frac{1}{(N+1)^2}-1-\frac{1}{2^2}\right)+$$
$$\lim_{N\to+\infty}3\left(\frac{1}{N}+\frac{1}{N+1}-\frac{2}{N}-\frac{2}{2}+1+\frac{1}{2}\right)=\frac{1}{16}.$$

A solution to the problem S263 of issue 2 2013 of Mathematical Reflections

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June 1, 2013

The Problem. *Proposed by Marcel Chirita, Bucharest, Romania*

Prove that for $n \geq 2$ *and* $1 \leq i \leq n$ *we have*

$$\sum_{j=1}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j}=1.$$

Solution : The problem, as stated, is not correct. We will show that for $n \geq 1$ and $1 \leq i \leq n$,

$$\sum_{j=1}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j}=\frac{(-1)^{n-1}}{i}.$$

It suffices to show that for $n \geq 1$ and $1 \leq i \leq n$,

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{\binom{n+j}{n}\binom{n}{n-j}}{i+j}=0,$$

or that for $n \geq 1$ and $0 \leq i \leq n-1$,

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{\binom{n+j}{n}}{n+j-i}=0.$$

Consider the operators S and I which act on the space $\mathbb{C}[x]$ of polynomials with complex coefficients with

$$p(x)\overset{S}{\mapsto}p(x+1)\quad\text{and}\quad p(x)\overset{I}{\mapsto}p(x),$$

i.e., the shifting and the identity operator respectively, and let us denote with Δ the forward difference operator, i.e.

$$\Delta := S - I \quad\text{with}\quad p(x)\overset{\Delta}{\mapsto}p(x+1)-p(x).$$

Note that for a non constant polynomial $p(x) \in \mathbb{C}[x]$ with $\deg(p(x)) = n$, we have $\deg(\Delta(p(x))) \leq n-1$, and for a constant polynomial $p(x) = c$ we have $\Delta(p(x)) = 0$.

Furthermore, since S and I clearly commute, from binomial theorem we have that for $n \geq 1$:

$$\Delta^n = (S - I)^n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} S^j I^{n-j},$$

so, for $n \geq 1$:

$$p(x) \overset{\Delta^n}{\mapsto} \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} p(x+j),$$

and, on account of the above, it is straightforward to see that Δ^n maps to the zero polynomial every polynomial with degree less than or equal to $n-1$.

Consider now for $n \geq 1$ and $0 \leq i \leq n-1$ the polynomial

$$p(x) = \frac{\binom{x}{n}}{x-i} = \frac{1}{n!} \frac{x(x-1)\cdots(x-n+1)}{x-i}.$$

Since $0 \leq i \leq n-1$, it is clear that $\deg\left(p(x)\right) = n-1$, so

$$\Delta^n\left(p(x)\right) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} p(x+j) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{\binom{x+j}{n}}{x+j-i} = 0$$

and hence

$$\Delta^n\left(p(x)\right)\Big|_{x=n} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{\binom{n+j}{n}}{n+j-i} = 0$$

as desired.

A solution to the problem U259 of issue 2 2013 of Mathematical Reflections

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May 3, 2013

The Problem. *Proposed by Arkady Alt, San Jose, California, USA*

Compute

$$\lim_{n\to+\infty}\frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}}.$$

Solution :

$$\begin{aligned}
\frac{\left(1+\frac{1}{n(n+a)}\right)^{n^3}}{\left(1+\frac{1}{n(n+b)}\right)^{n^3}} &= \left(\frac{1+\frac{1}{n^2(1+a/n)}}{1+\frac{1}{n^2(1+b/n)}}\right)^{n^3} = \left(\frac{1+\frac{1}{n^2}\left(1-\frac{a}{n}+\mathcal{O}(n^{-2})\right)}{1+\frac{1}{n^2}\left(1-\frac{b}{n}+\mathcal{O}(n^{-2})\right)}\right)^{n^3}\\
&= \left(\frac{1+\frac{1}{n^2}-\frac{a}{n^3}+\mathcal{O}(n^{-4})}{1+\frac{1}{n^2}-\frac{b}{n^3}+\mathcal{O}(n^{-4})}\right)^{n^3}\\
&= \left(\left(1+\frac{1}{n^2}-\frac{a}{n^3}+\mathcal{O}(n^{-4})\right)\left(1+\frac{1}{n^2}-\frac{b}{n^3}+\mathcal{O}(n^{-4})\right)^{-1}\right)^{n^3}\\
&= \left(\left(1+\frac{1}{n^2}-\frac{a}{n^3}+\mathcal{O}(n^{-4})\right)\left(1-\frac{1}{n^2}+\frac{b}{n^3}+\mathcal{O}(n^{-4})\right)\right)^{n^3}\\
&= \left(1+\frac{b-a}{n^3}+\mathcal{O}(n^{-4})\right)^{n^3}\\
&= \exp\left(n^3\ln\left(1+\frac{b-a}{n^3}+\mathcal{O}(n^{-4})\right)\right)\\
&= \exp\left(n^3\left(\frac{b-a}{n^3}+\mathcal{O}(n^{-4})\right)\right)\\
&= \exp\left(b-a+\mathcal{O}(n^{-1})\right) = e^{b-a}+\mathcal{O}(n^{-1}) \to e^{b-a}.
\end{aligned}$$

A solution to the problem U262 of issue 2 2013 of Mathematical Reflections

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May 3, 2013

The Problem. *Proposed by Ivan Borsenco, Massachusetts Institute of Technology, USA*

Let a *and* b *be positive real numbers. Find* $\lim_{n\to+\infty}\sqrt[n]{\prod_{i=1}^{n}\left(a+\frac{b}{i}\right)}$.

Solution : We have

$$\begin{aligned}\sqrt[n]{\prod_{i=1}^{n}\left(a+\frac{b}{i}\right)} &= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\ln\left(a+\frac{b}{i}\right)\right)\\ &= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\left(\ln a+\ln\left(1+\frac{b}{ai}\right)\right)\right)\\ &= a\exp\left(\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{b}{ai}\right)\right)\end{aligned}$$

but from Cesàro Stolz theorem,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{b}{ai}\right)=\lim_{n\to+\infty}\ln\left(1+\frac{b}{an}\right)=0,$$

so by the continuity of e^x the desired limit is a.

A solution to the problem U264 of issue 2 2013 of Mathematical Reflections

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May 3, 2013

The Problem. *Proposed by Mihai Piticari, "Dragos Voda" National College, Romania*

Let A *be a finite ring such that* $1+1=0$*. Prove that the equations* $x^2=0$ *and* $x^2=1$ *have the same number of solutions in* A*.*

Solution : Let us denote with $A_1 \subseteq A$ the set of solutions of $x^2=0$ in A and with $A_2 \subseteq A$ the set of solutions of $x^2=1$ in A.

We have

$$a \in A_1 \Rightarrow a^2 = 1+1 \Rightarrow a^2 - 1 = 1 \Rightarrow (a-1)(a+1) = 1 \overset{1=-1}{\Longrightarrow} (a-1)^2 = 1$$

and, since $a-1 \in A$, we conclude that A_1 can be injected in A_2.

Conversely,

$$a \in A_2 \Rightarrow a^2 - 1 = 0 \Rightarrow (a-1)(a+1) = 0 \overset{1=-1}{\Longrightarrow} (a-1)^2 = 0$$

and, since $a-1 \in A$, we conclude that A_2 can be injected in A_1.

Since A is finite we get the desired result.

A solution to the problem U207 of issue 5, 2011 of Mathematical Reflections

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Wednesday 28/9/2011

• **The Problem :** Evaluate $\displaystyle\sum_{n=1}^{\frac{n-1}{2}} \sec\left(\frac{2k\pi}{n}\right)$.

◇ **Solution :** By De Moivre's formula we have

$$\cos(n\theta) = \mathfrak{Re}\left[(\cos\theta + i\sin\theta)^n\right] = \mathfrak{Re}\left[\sum_{k=0}^{n}\binom{n}{k}\cos^k\theta\cdot(i\cdot\sin\theta)^{n-k}\right] =$$

$$\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(-1)^k(\cos\theta)^{n-2k}(\sin\theta)^{2k} = \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(\cos\theta)^{n-2k}(\cos^2\theta-1)^k.$$

Since n is odd,

$$\cos(n\theta)-1 = \sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2k}(\cos\theta)^{n-2k}(\cos^2\theta-1)^k - 1 = a_n\cos^n\theta + a_{n-1}\cos^{n-1}\theta + \ldots + a_1\cos\theta - 1$$

has as its roots $k\dfrac{2\pi}{n}$, $\quad k = 1, 2, \ldots, \frac{n-1}{2}$ with multiplicity two, and 0, and hence

$$P(x) := \sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2k}x^{n-2k}(x^2-1)^k - 1 = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x - 1$$

has as its roots $\cos\left(\dfrac{2k\pi}{n}\right)$, $\quad k = 1, 2, \ldots, \frac{n-1}{2}$ with multiplicity two, and 1.

It follows that

$$Q(x) := \frac{P(x)}{x-1} = a_nx^{n-1} + (a_n + a_{n-1})x^{n-2} + \ldots + (a_n + \ldots + a_2)x + (a_n + \ldots + a_1)$$

has $\cos\left(\dfrac{2k\pi}{n}\right)$, $\quad k = 1, 2, \ldots, \frac{n-1}{2}$ with multiplicity two as its roots, so

$\sec\left(\dfrac{2k\pi}{n}\right)$, $\quad k = 1, 2, \ldots, \frac{n-1}{2}$ with multiplicity two are the roots of

$$\begin{aligned}
x^{n-1}Q\left(\frac{1}{x}\right) &= (a_n + \ldots + a_1)x^{n-1} + (a_n + \ldots + a_2)x^{n-2} + \ldots + (a_n + a_{n-1})x + a_n \\
&= (P(1)+1)x^{n-1} + (P(1)+1-a_1)x^{n-2} + \ldots + (a_n + a_{n-1})x + a_n \\
&= x^{n-1} + (1-a_1)x^{n-2} + \ldots + (a_n + a_{n-1})x + a_n.
\end{aligned}$$

But since n is odd, $a_1 = \lim_{\theta \to \frac{\pi}{2}} \frac{\cos(n\theta)}{\cos\theta} \overset{DLH}{=} n(-1)^{\frac{n-1}{2}}$, so

$$\sum_{k=1}^{\frac{n-1}{2}} \sec\left(\frac{2k\pi}{n}\right) = \frac{1}{2}\left(-\frac{1-a_1}{2}\right) = \frac{a_1-1}{2} = \frac{n(-1)^{\frac{n-1}{2}}-1}{2},$$

the $\frac{1}{2}$ factor being justified by the multiplicity of the roots of $x^{n-1}Q\left(\frac{1}{x}\right)$.

A solution to the problem O243 of issue 5 2012 of Mathematical Reflections

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October 21, 2012

The Problem : Proposed by Iurie Boreico, Stanford University, USA.

Let m, n *be positive numbers with* $n > m$. *Prove that*

$$\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{m+n-2k}{n-1}=\binom{n}{m+1}.$$

Solution : By Cauchy's theorem we have that $\binom{n}{m}=\frac{1}{2\pi i}\int_R\frac{(z+1)^n}{z^{m+1}}\,dz$, where R is any circle surrounding the origin.

Now:

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{m+n-2k}{n-1} &= \frac{1}{2\pi i}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\int_R\frac{(z+1)^{m+n-2k}}{z^n}\,dz\\
&= \frac{1}{2\pi i}\int_R\frac{(z+1)^{m+n}}{z^n}\sum_{k=0}^{n}(-1)^n\binom{n}{k}\frac{1}{(z+1)^{2k}}\,dz\\
&= \frac{1}{2\pi i}\int_R\frac{(z+1)^{m+n}}{z^n}\left(1-\frac{1}{(z+1)^2}\right)^n dz\\
&= \frac{1}{2\pi i}\int_R\frac{(z+2)^n}{(z+1)^{n-m}}\,dz\\
{}^1 &= \mathrm{Res}_{z=-1}\frac{(z+2)^n}{(z+1)^{n-m}}\\
&= \lim_{z\to-1}\frac{1}{(n-m-1)!}\frac{d^{n-m-1}}{dz^{n-m-1}}\left((z+2)^n\right)\\
&= \binom{n}{m+1}
\end{aligned}$$

[1] by Cauchy's residue theorem, since the integrand has a pole of order $n-m$ at $z=-1$.

Implicit function : Proposed problem for the School Science and Mathematics journal

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June 3, 2013

The Problem : *Let* $x \geq \frac{1+\ln 2}{2}$ *and* $f(x)$ *be the function defined by the relations :*

$$f^2(x) - \ln f(x) = x \tag{1}$$

$$f(x) \geq \frac{\sqrt{2}}{2}. \tag{2}$$

1. Calculate the limit $\lim\limits_{x\to+\infty} \dfrac{f(x)}{\sqrt{x}}$, if it exists.
2. Find the values of $\alpha \in \mathbb{R}$ for which the series $\sum\limits_{k=1}^{+\infty} k^{\alpha}\left(f(k) - \sqrt{k}\right)$ converges.
3. Calculate the limit $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x}f(x) - x}{\ln x}$, if it exists.

Solution :

Define at first the symbol $\sim$ by $f_1(x) \sim f_2(x) \Leftrightarrow \lim\limits_{x\to+\infty} \dfrac{f_1(x)}{f_2(x)} = 1$ for eventually non-vanishing functions.

We can easily see that the function $g(x) := x^2 - \ln x$ is strictly increasing for $x \geq \dfrac{\sqrt{2}}{2}$ and $\lim\limits_{x\to+\infty} g(x) = +\infty$. This means that $f(x)$ has the same properties, so we can write

$$x = f^2(x) - \ln f(x) \sim f^2(x), \quad \text{so} \quad x \sim f^2(x), \quad \text{hence} \quad \frac{f^2(x)}{x} = 1 + o(1). \tag{3}$$

Furthermore, (1) gives

$$\ln f(x) = f^2(x) - x \Rightarrow \frac{\ln f(x)}{x} = \frac{f^2(x)}{x} - 1 \overset{(3)}{\Rightarrow} \frac{\ln f(x)}{x} = o(1), \tag{4}$$

so for x big enough we have

$$f(x) = (x + \ln f(x))^{1/2} = \sqrt{x}\left(1 + \frac{\ln f(x)}{x}\right)^{1/2} \overset{(4)}{=} \sqrt{x}(1 + o(1))^{1/2} = \sqrt{x}(1 + o(1)). \tag{5}$$

Now

$$\begin{aligned} f^2(x) &= x + \ln f(x) \overset{(5)}{=} x + \ln\left(\sqrt{x}(1 + o(1))\right) = x + \mathcal{O}(\ln x) \Rightarrow \\ f(x) &= (x + \mathcal{O}(\ln x))^{1/2} = \sqrt{x}\left(1 + \mathcal{O}(x^{-1}\ln x)\right)^{1/2} \\ &= \sqrt{x} + \mathcal{O}(x^{-1/2}\ln x). \end{aligned} \tag{6}$$

Plugging (6) into (1) again we get

$$\begin{aligned} f^2(x) &= x + \ln\left(\sqrt{x} + \mathcal{O}(x^{-1/2}\ln x)\right) = x + \frac{\ln x}{2} + \mathcal{O}(x^{-1}\ln x) \Rightarrow \\ f(x) &= \sqrt{x}\left(1 + \frac{\ln x}{2x} + \mathcal{O}(x^{-2}\ln x)\right)^{1/2} \\ &= \sqrt{x}\left(1 + \frac{\ln x}{4x} + \mathcal{O}(x^{-2}\ln^2 x)\right) \\ &= \sqrt{x} + \frac{\ln x}{4\sqrt{x}} + \mathcal{O}(x^{-3/2}\ln^2 x). \end{aligned} \tag{7}$$

From the above :

1. we directly see that $\displaystyle\lim_{x\to+\infty}\frac{f(x)}{\sqrt{x}} = 1$,

2. $k^a\left(f(k) - \sqrt{k}\right) = \dfrac{k^{a-1/2}\ln k}{4} + \mathcal{O}(k^{a-3/2}\ln^2 k)$, so by the integral test for convergence, since

$$\int_1^{+\infty} x^{\alpha-1/2}\ln x\,dx \overset{\ln x = t}{=} \int_0^{+\infty} te^{(1/2+\alpha)t}\,dt = \begin{cases} +\infty & ,\alpha \geq -\frac{1}{2} \\ \in \mathbb{R} & ,\alpha < -\frac{1}{2} \end{cases},$$

   the same holds for the series $\displaystyle\sum_{k=1}^{+\infty} k^a\left(f(k) - \sqrt{k}\right)$,

3. $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{x}f(x) - x}{\ln x} = \frac{1}{4}$.

A solution to the problem 5208 of April's 2012 issue of School Science and Mathematics journal

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April 20, 2012

The Problem : Proposed by D. M. Bătinetu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

Let the sequence of positive real numbers $\{a_n\}_{n\geq 1}, \quad n \in \mathbb{Z}^+$ *be such that* $\lim\limits_{n\to+\infty} \frac{a_{n+1}}{n^2 a_n} = b$. *Calculate*

$$\lim_{n\to+\infty} \frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n}.$$

Solution : Setting $z_n := \frac{a_n}{n^{2n}}$, we have

$$\frac{z_{n+1}}{z_n} = \frac{a_{n+1}}{n^2 a_n}\left[\left(1+\frac{1}{n}\right)^n\right]^{-2}\left(1+\frac{1}{n}\right)^{-2} \to be^{-2}. \tag{1}$$

and by Cesàro Stolz :

$$\begin{aligned}\lim_{n\to+\infty} z_n^{1/n} &= \exp\left(\lim_{n\to+\infty}\frac{\ln z_n}{n}\right)\\ &= \exp\left(\lim_{n\to+\infty}\ln\frac{z_{n+1}}{z_n}\right)\\ &= \exp\left(\ln\lim_{n\to+\infty}\frac{z_{n+1}}{z_n}\right)\\ &= be^{-2}. \end{aligned}\tag{2}$$

On account of (1) and (2) :

$$\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{n z_n^{1/n}}\right)^n = \left(1+\frac{1}{n}\right)^n \frac{z_{n+1}}{z_n} z_{n+1}^{-\frac{1}{n+1}} \to e,$$

so

$$\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}=z_n^{1/n}\left(\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)^n\right)\to be^{-2},$$

since

$$\begin{aligned}\lim_{n\to+\infty}\frac{\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}&=\lim_{n\to+\infty}\frac{\exp\left(\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)\right)-1}{\ln\left(\frac{(n+1)z_{n+1}^{\frac{1}{n+1}}}{nz_n^{1/n}}\right)}\\&=\lim_{x\to0}\frac{e^x-1}{x}=1.\end{aligned}$$

A solution to the problem 5211 of April's 2012 issue of School Science and Mathematics journal

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April 28, 2012

The Problem : Proposed by Ovidiu Furdui, Cluj–Napoca, Romania

Let $n \geq 1$ *be a natural number and let* $f_n(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$*, where the number of x's in the definition of* f_n *is* n*. For example*

$$f_1(x) = x, \quad f_2(x) = x^x, \quad f_3(x) = x^{x^x}, \dots .$$

Calculate the limit

$$\lim_{x \to 1} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^{n+1}}.$$

Solution : We easily see by induction that

$$f_n(x) = 1 + \mathcal{O}(1-x) \qquad \text{for } n \geq 1. \tag{1}$$

Now we will show, by induction again, that

$$f_n(x) = f_{n-1}(x) + (-1)^n(1-x)^n + \mathcal{O}\left((1-x)^{n+1}\right) \qquad \text{for } n \geq 2. \tag{2}$$

Indeed,

- for $n = 2$:

$$\begin{aligned} f_2(x) = x^x &= x^{1-(1-x)} \\ &= x \exp\left(-(1-x)\ln\left(1-(1-x)\right)\right) \\ &= x \exp\left((1-x)^2 + \mathcal{O}\left((1-x^3)\right)\right) \\ &= x\left(1 + (1-x)^2 + \mathcal{O}\left((1-x)^3\right)\right) \\ &= x + (-1)^2(1-x)^2 + \mathcal{O}\left((1-x)^3\right) \\ &= f_1(x) + (-1)^2(1-x)^2 + \mathcal{O}\left((1-x)^3\right). \end{aligned}$$

- If for $n = k \geq 2$

$$f_k(x) = f_{k-1}(x) + (-1)^k(1-x)^k + \mathcal{O}\left((1-x)^{k+1}\right) \quad \text{is true, then}$$

- for $n = k+1$ we have

$$\begin{aligned}
f_{k+1}(x) &= x^{f_k(x)} \\
&= x^{f_{k-1}(x)+(-1)^k(1-x)^k+\mathcal{O}\left((1-x)^{k+1}\right)} \\
&= x^{f_{k-1}(x)} \cdot x^{(-1)^k(1-x)^k+\mathcal{O}\left((1-x)^{k+1}\right)} \\
&= f_k(x) \exp\left(\left((-1)^k(1-x)^k + \mathcal{O}\left((1-x)^{k+1}\right)\right) \ln\left(1-(1-x)\right)\right) \\
&= f_k(x) \exp\left(\left((-1)^k(1-x)^k + \mathcal{O}\left((1-x)^{k+1}\right)\right) \left(-(1-x) + \mathcal{O}\left((1-x)^2\right)\right)\right) \\
&= f_k(x) \exp\left((-1)^{k+1}(1-x)^{k+1} + \mathcal{O}\left((1-x)^{k+2}\right)\right) \\
&= f_k(x) \left(1 + (-1)^{k+1}(1-x)^{k+1} + \mathcal{O}\left((1-x)^{k+2}\right)\right) \\
&= f_k(x) + f_k(x)(-1)^{k+1}(1-x)^{k+1} + \mathcal{O}\left(f_k(x)(1-x)^{k+2}\right) \\
&\overset{(1)}{=} f_k(x) + (-1)^{k+1}(1-x)^{k+1} + \mathcal{O}\left((1-x)^{k+2}\right).
\end{aligned}$$

From (2) we get that for $n \geq 2$:

$$\begin{aligned}
&\lim_{n\to+\infty} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^n} = (-1)^n && \text{and} \\
&\lim_{n\to+\infty} \frac{f_n(x) - f_{n-1}(x)}{(1-x)^{n+1}} && \text{doesn't exist.}
\end{aligned}$$

A solution to the problem 5185 of December's 2011 issue of School Science and Mathematics journal

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April 22, 2012

The Problem : *Proposed by José Luis Díaz-Barrero, Barcelona, Spain*

Calculate, without using a computer, the value of

$$\sin\left(\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right).$$

Solution :

The following identities are well known :

$$\arctan a+\arctan b=\begin{cases}\arctan\frac{a+b}{1-ab} & ,ab<1\\ \arctan\frac{a+b}{1-ab}+\pi & ,ab>1\wedge a>0\\ \arctan\frac{a+b}{1-ab}-\pi & ,ab>1\wedge a<0\end{cases}\tag{1}$$

$$\arctan a+\arctan\frac{1}{a}=\begin{cases}\frac{\pi}{2} & ,a>0\\ -\frac{\pi}{2} & ,a<0\end{cases}\tag{2}$$

Applying (1) to the pair $\arctan\left(\frac{1}{3}\right),\arctan\left(\frac{1}{5}\right)$ and repeating to $\arctan\left(\frac{1}{7}\right)$, $\arctan\left(\frac{1}{11}\right)$ and $\arctan\left(\frac{1}{13}\right)$, after trivial calculations we get

$$\sin\left(\arctan\left(\frac{1}{3}\right)+\arctan\left(\frac{1}{5}\right)+\arctan\left(\frac{1}{7}\right)+\arctan\left(\frac{1}{11}\right)+\arctan\left(\frac{1}{13}\right)+\arctan\left(\frac{111}{121}\right)\right)=$$

$$\sin\left(\arctan\left(\frac{121}{111}\right)+\arctan\left(\frac{111}{121}\right)\right)\overset{(2)}{=}\sin\frac{\pi}{2}=1.$$

A solution to the problem 5198 of February's 2012 issue of School Science and Mathematics journal

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April 23, 2012

The Problem : Proposed by José Luis Díaz-Barrero, Barcelona, Spain.

Let m, n *be positive integers. Calculate,*

$$\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\left[\frac{k+1}{2}\right]+a+i\right)^{-1},$$

where a *is a nonnegative number and* $[x]$ *represents the greatest integer less than or equal to* x.

Solution : By a direct calculation, using the identity $\Gamma(x+1)=x\Gamma(x), \quad x>0$ for the Γ function, we can see that

$$\prod_{i=0}^{m}\frac{1}{b+i}=\frac{\Gamma(b)}{\Gamma(b+m+1)}=\frac{1}{m}\left(\frac{\Gamma(b)}{\Gamma(b+m)}-\frac{\Gamma(b+1)}{\Gamma(b+m+1)}\right) \qquad b>0. \tag{1}$$

Now

$$\begin{aligned}
\sum_{k=1}^{2n}\prod_{i=0}^{m}\left(\left[\frac{k+1}{2}\right]+a+i\right)^{-1} &= \sum_{k=1,3,\dots,2n-1}\prod_{i=0}^{m}\left(\frac{k+1}{2}+a+i\right)^{-1}+\sum_{k=2,4,\dots,2n}\prod_{i=0}^{m}\left(\frac{k}{2}+a+i\right)^{-1}\\
&=2\sum_{k=1}^{n}\prod_{i=0}^{m}(k+a+i)^{-1}\\
&\overset{(1)}{=}\frac{2}{m}\sum_{k=1}^{n}\left(\frac{\Gamma(a+k)}{\Gamma(a+k+m)}-\frac{\Gamma(a+k+1)}{\Gamma(a+k+m+1)}\right)\\
&=\frac{2}{m}\left(\frac{\Gamma(a+1)}{\Gamma(a+1+m)}-\frac{\Gamma(a+n+1)}{\Gamma(a+n+m+1)}\right).
\end{aligned}$$

A solution to the problem 5199 of February's 2012 issue of School Science and Mathematics journal

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April 23, 2012

The Problem : *Proposed by Ovidiu Furdui, Cluj, Romania*

Let $k > 0$ and $n \geq 0$ be real numbers. Calculate,

$$I := \int_0^1 x^n \ln\left(\sqrt{1+x^k} - \sqrt{1-x^k}\right) dx.$$

Solution : We got

$$\begin{aligned}
I &= \left.\frac{x^{n+1}\ln\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)}{n+1}\right|_0^1 - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\frac{1}{\sqrt{1+x^k}}+\frac{1}{\sqrt{1-x^k}}\right)}{\sqrt{1+x^k}-\sqrt{1-x^k}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \frac{x^{n+k}\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)}{\left(\sqrt{1+x^k}-\sqrt{1-x^k}\right)\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{4(n+1)}\int_0^1 \frac{x^n\left(\sqrt{1-x^k}+\sqrt{1+x^k}\right)^2}{\sqrt{1-x^{2k}}}\,dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)}\int_0^1 \left(\frac{x^n}{\sqrt{1-x^{2k}}}+x^n\right)dx \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{k}{2(n+1)}\int_0^1 \frac{x^n}{\sqrt{1-x^{2k}}}\,dx \\
&\overset{x^{2k}=u}{=\!=} \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}\mathrm{B}\left(\frac{n+1}{2k},\frac{1}{2}\right) \\
&= \frac{\ln 2}{2(n+1)} - \frac{k}{2(n+1)^2} - \frac{1}{4(n+1)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2k}\right)}{\Gamma\left(\frac{n+k+1}{2k}\right)}
\end{aligned}$$

A solution to the problem 5242 of February's 2013 issue of School Science and Mathematics journal

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February 15, 2013

The Problem. *Proposed by Kenneth Korbin, New York, NY*

Let N *be any positive integer, and let* $x = N(N+1)$. *Find the value of*

$$\sum_{K=0}^{x/2} \binom{x-K}{K} x^K.$$

Solution : Using m instead of x for notation convenience we compute the generating function of $\sum_{k=0}^{m/2} \binom{m-k}{k} y^k$:

$$\begin{aligned}\sum_{m\geq 0}\sum_{k=0}^{m/2}\binom{m-k}{k}y^k t^m &= \sum_{k\geq 0} y^k \sum_{m\geq 2k}\binom{m-k}{k}t^m = \sum_{k\geq 0} y^k \sum_{m\geq 0}\binom{m+k}{k}t^{m+2k}\\ &= \sum_{k\geq 0} y^k \sum_{m\geq 0}\binom{m+k}{m}t^{m+2k} = \sum_{k\geq 0}(yt^2)^k \sum_{m\geq 0}\binom{-k-1}{m}(-t)^m\\ &= \sum_{k\geq 0}(yt^2)^k(1-t)^{-k-1} = \frac{1}{1-t}\sum_{k\geq 0}\left(\frac{yt^2}{1-t}\right)^k\\ &= \frac{1}{1-t-yt^2}\end{aligned}$$

It is easily shown, decomposing into partial fraction and expanding the geometric series, that if $ax^2 + by + c$ has two distinct non negative roots ρ_1, ρ_2, then

$$\frac{1}{ax^2+by+c} = \sum_{m\geq 0}\frac{1}{a(\rho_1-\rho_2)}\left(\rho_2^{-m-1}-\rho_1^{-m-1}\right)x^m,$$

so

$$\sum_{m\geq 0}\sum_{k=0}^{m/2}\binom{m-k}{k}y^k t^m=\sum_{m\geq 0}\frac{1}{\sqrt{1+4y}}\left(\left(\frac{-2y}{1-\sqrt{1+4y}}\right)^{m+1}-\left(\frac{-2y}{1+\sqrt{1+4y}}\right)^{m+1}\right)t^m$$

and hence

$$\sum_{k=0}^{m/2}\binom{m-k}{k}y^k=\frac{1}{\sqrt{1+4y}}\left(\left(\frac{-2y}{1-\sqrt{1+4y}}\right)^{m+1}-\left(\frac{-2y}{1+\sqrt{1+4y}}\right)^{m+1}\right).$$

Putting m in the place of y and then $N(N+1)$ in the place of m in the above relation, and since $N(N+1)+1$ is odd, we get

$$\sum_{K=0}^{N(N+1)/2}\binom{N(N+1)-K}{K}(N(N+1))^K=\frac{1}{2N+1}\left((N+1)^{N^2+N+1}+N^{N^2+N+1}\right).$$

A solution to the problem 5247 of February's 2013 issue of School Science and Mathematics journal

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February 15, 2013

The Problem. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\lim_{n\to+\infty}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx}.$$

Solution : For $n\in\mathbb{N}$, $x\in(0,1]$ we have

$$\begin{aligned}\ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx}) &= n!x^n\prod_{k=1}^{n}\left(1+\frac{\ln(1+e^{-kx})}{kx}\right)=n!x^n\prod_{k=1}^{n}\left(1+\mathcal{O}\left(\frac{e^{-kx}}{kx}\right)\right)\\ &= n!x^n\left(1+\mathcal{O}\left(\frac{e^{-x}}{x^n}\right)\right)\\ &= n!\left(x^n+\mathcal{O}\left(e^{-x}\right)\right)\end{aligned}$$

so

$$\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})=\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right)$$

Now from the above and taking into account that, from Stirling's formula,

$$\ln n! = n\ln n - n + \mathcal{O}(\ln n)$$

we get that

$$\begin{aligned}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx} &= \frac{1}{n}\exp\left(\frac{1}{n}\ln\left(\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right)\right)\right)\\ &= \frac{1}{n}\exp\left(\ln n - 1+\mathcal{O}\left(\frac{\ln n}{n}\right)\right)=e^{-1}+\mathcal{O}\left(\frac{\ln n}{n}\right)\to e^{-1}.\end{aligned}$$

A solution to the problem 5193 of January's 2012 issue of School Science and Mathematics journal

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April 28, 2012

The Problem : *Proposed by Proposed by Ovidiu Furdui, Cluj-Napoca, Romania.*

Let f be a function which has a power series expansion at 0 with radius of convergence R.

1. Prove that $\displaystyle\sum_{n=1}^{+\infty} nf^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right) = \int_0^x e^{x-t} t f'(t)\,dt,$ $\quad |x| < R,$
2. Let α be a non-zero real number. Calculate $\displaystyle\sum_{n=1}^{+\infty} n\alpha^n\left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right).$

Solution :

1. From the problem's assumptions we have that

$$f(x) = \sum_{n=0}^{+\infty}\frac{f^{(n)}(0)}{n!}x^n \quad \text{and} \quad f'(x) = \sum_{n=1}^{+\infty}\frac{f^{(n)}(0)}{(n-1)!}x^{n-1} \quad \text{for} \quad |x| < R,$$

so, for $|x| < R$ we got

$$\begin{aligned}\int_0^x e^{x-t} t f'(t)\,dt &= \int_0^x e^{x-t}\sum_{n=1}^{+\infty}\frac{f^{(n)}(0)}{(n-1)!}t^n\,dt \\ &= e^x\sum_{n=1}^{+\infty}\frac{f^{(n)}(0)}{(n-1)!}\int_0^x t^n e^{-t}\,dt \\ &:= e^x\sum_{n=1}^{+\infty}\frac{f^{(n)}(0)}{(n-1)!}I_n. \end{aligned} \tag{1}$$

Now $I_n = -\int_0^x t^n (e^{-t})'\,dt = -x^n e^{-x} + nI_{n-1}$, so it is easily verified by induction that

$$I_n = -e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!,$$

With the above, (1) will give

$$\begin{aligned}
\int_0^x e^{x-t} t f'(t)\,dt &= e^x \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(-e^{-x}\left(x^n + nx^{n-1} + \cdots + n!x^0\right) + n!\right) \\
&= \sum_{n=1}^{+\infty} \frac{f^{(n)}(0)}{(n-1)!}\left(n!e^x - x^n - nx^{n-1} - \cdots - n!x^0\right) \\
&= \sum_{n=1}^{+\infty} n f^{(n)}(0)\left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right).
\end{aligned}$$

2. From 1 with $f(x) = e^{\alpha x}$ we got that

$$\begin{aligned}
\sum_{n=1}^{+\infty} n\alpha^n \left(e^x - 1 - \frac{x}{1!} - \cdots - \frac{x^n}{n!}\right) &= \int_0^x e^{x-t}\alpha t e^{\alpha t}\,dt \\
&:= I_\alpha.
\end{aligned}$$

Now

- for $\alpha = 1$ it is $I_1 = \displaystyle\int_0^x e^{x-t} t e^t\,dt = \frac{x^2 e^x}{2}$ and
- for $\alpha \neq 1$ it is

$$\begin{aligned}
I_\alpha &= \alpha e^x \left(\int_0^x t\left(\frac{e^{(\alpha-1)t}}{\alpha-1}\right)' dt\right) \\
&= \frac{\alpha e^{\alpha x}}{\alpha-1}\left(x - \frac{1}{\alpha-1}\right) + \frac{\alpha e^x}{(\alpha-1)^2}.
\end{aligned}$$

A solution to the problem 5241 of January's 2013 issue of School Science and Mathematics journal

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January 23, 2013

The Problem. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a \geq 0$ *be a real number. Calculate*

$$\lim_{n \to +\infty} \left(\int_0^1 \sqrt[n]{x^n + a}\, dx \right)^n .$$

Solution :

1. For $a = 0$ the limit is trivially $0 = a$.
2. For $a > 0$: We set $I_n^n := \left(\int_0^1 \sqrt[n]{x^n + a}\, dx \right)^n = \exp\left(n \ln \left(\int_0^1 \sqrt[n]{x^n + a}\, dx \right) \right) := e^{A_n}$.

   Now, considering that $n \in [1, +\infty)$, since $0 < \sqrt[n]{x^n + a} \leq 1 + a$ and $\sqrt[n]{x^n + a} \overset{n \to +\infty}{\longrightarrow} \mathbf{1}$ for $x \in [0, 1]$, by dominated convergence theorem we get that $I_n \to 1$, thus $\ln I_n \to 0$.

   Furthermore, by Leibniz's rule we have that for $n \geq 1$

   $$\frac{\partial I_n}{\partial n} = \int_0^1 \frac{\partial}{\partial n} \sqrt[n]{x^n + a}\, dx = \int_0^1 (x^n + a)^{\frac{1-n}{n}} \left(\frac{n x^n \ln x - (x^n + a) \ln(x^n + a)}{n^2} \right) dx.$$

   We also have that

   $$\begin{aligned} \left| (x^n + a)^{\frac{1-n}{n}} \left((x^n + a) \ln(x^n + a) - n x^n \ln x \right) \right| &\leq \frac{1 + a}{a} \left(|(x^n + a) \ln(x^n + a)| + |n x^n \ln x| \right) \\ &\leq \frac{1 + a}{a} \left(\max\{e^{-1}, (1 + a) \ln(1 + a)\} + e^{-1} \right) \end{aligned}$$

and since $(x^n+a)^{\frac{1-n}{n}}\left((x^n+a)\ln(x^n+a)-nx^n\ln x\right)\to\begin{cases}\ln(1+a) & ,x=1\\ \ln a & ,x\in[0,1)\end{cases}$, by the dominated convergence theorem it is $-n^2\dfrac{\partial I_n}{\partial n}\to\ln a$.

Now applying De l' Hospital's rule we get

$$\lim_{n\to+\infty}A_n=\lim_{n\to+\infty}\frac{\ln I_n}{n^{-1}}=\lim_{\mathbb{R}\ni n\to+\infty}\frac{\ln I_n}{n^{-1}}\overset{0/0}{=}\lim_{n\to+\infty}I_n^{-1}\cdot\left(-n^2\frac{\partial I_n}{\partial n}\right)\to\ln a,$$

so the required limit in each case is a.

A solution to the problem 5203 of March's 2012 issue of School Science and Mathematics journal

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April 28, 2012

The Problem : *Proposed by Pedro Pantoja, Natal–RN, Brazil*

Evaluate,

$$I := \int_0^{\pi/4} \ln\left(\frac{1+\sin^2(2x)}{\sin^4 x+\cos^4 x}\right) dx.$$

Solution : Using the identities

$$\sin^2 x = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1+\cos(2x)}{2},$$

we easily get that

$$\frac{1+\sin^2(2x)}{\sin^4 x+\cos^4 x} = 2\frac{3-\cos(4x)}{3+\cos(4x)}, \quad \text{so}$$

$$\begin{aligned} I &= \int_0^{\pi/4} \ln(2) + \ln\left(\frac{3-\cos(4x)}{3+\cos(4x)}\right) dx \overset{4x=y}{=\!=\!=} \frac{\pi\ln 2}{4} + \frac{1}{4}\int_0^{\pi} \ln\left(\frac{3-\cos y}{3+\cos y}\right) dy \\ &:= \frac{\pi\ln 2}{4} + \frac{1}{4}J. \end{aligned} \tag{1}$$

Now

$$J \overset{\pi-y=t}{=\!=\!=} \int_0^{\pi} \ln\left(\frac{3+\cos t}{3-\cos t}\right) dt = -J \quad \text{so} \quad J = 0,$$

and with this, (1) will give

$$I = \frac{\pi\ln 2}{4}.$$

A solution to the problem 5205 of March's 2012 issue of School Science and Mathematics journal

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April 28, 2012

The Problem : *Proposed by Ovidiu Furdui, Cluj–Napoca, Romania*

Find the sum,

$$\sum_{n=1}^{+\infty}\left(1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{(-1)^{n-1}}{n}-\ln 2\right)\cdot\ln\left(\frac{n+1}{n}\right).$$

Solution : We set

$$f_m(x)=\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\qquad x<1,$$

so we ask to find

$$\lim_{m\to+\infty}f_m(-1).$$

For $x<1$ we have

$$\begin{aligned}
f_m'(x)&=\left(\sum_{n=1}^{m}\left(-\sum_{k=1}^{n}\frac{x^k}{k}-\ln(1-x)\right)\ln\left(\frac{n+1}{n}\right)\right)'\\
&=\sum_{n=1}^{m}\left(-\sum_{k=0}^{n-1}x^k+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&=\sum_{n=1}^{m}\left(-\frac{1-x^n}{1-x}+\frac{1}{1-x}\right)\ln\left(\frac{n+1}{n}\right)\\
&=\frac{1}{1-x}\sum_{n=1}^{m}x^n\left(\ln(n+1)-\ln n\right)\\
&=\frac{1}{1-x}\left(\sum_{n=2}^{m}\left(x^{n-1}-x^n\right)\ln n+x^m\ln(m+1)\right)\\
&=\sum_{n=2}^{m}x^{n-1}\ln n+\frac{x^m}{1-x}\ln(m+1)
\end{aligned}$$

so we integrate from 0 to y, where $y<1$, to get

$$f_m(y)=\sum_{n=2}^{m}\frac{y^n}{n}\ln n+\ln(m+1)\int_0^y\frac{x^m}{1-x}\,dx$$

and set $y=-1$ to get

$$\begin{aligned}f_m(-1)&=\sum_{n=2}^{m}\frac{(-1)^n}{n}\ln n+\ln(m+1)\int_0^{-1}\frac{x^m}{1-x}\,dx\\&\overset{x=-t}{=\!=\!=}\sum_{n=2}^{m}\frac{(-1)^n}{n}\ln n+(-1)^{m+1}\ln(m+1)\int_0^1\frac{t^m}{1+t}\,dt\\&:=A_m+(-1)^{m+1}\ln(m+1)B_m.\end{aligned}\tag{1}$$

Now integrating by parts,

$$\begin{aligned}B_m&=\left.\frac{t^{m+1}}{(m+1)(1+t)}\right|_0^1+\frac{1}{m+1}\int_0^1\frac{t^{m+1}}{(1+t)^2}\,dt\\&\leq\frac{1}{2(m+1)}+\frac{1}{m+1}\int_0^1\frac{1}{(1+t)^2}\,dt\\&=\frac{1}{m+1}<\frac{1}{m}\end{aligned}\tag{2}$$

and for A_m, since it converges from Dirichlet's Criterion, we can write

$$\lim_{m\to+\infty}A_m=\lim_{m\to+\infty}A_{2m}$$

and

$$
\begin{aligned}
A_{2m} &= \sum_{n=1}^{2m} \frac{(-1)^n}{n} \ln n \\
&= \sum_{n=1}^{m} \frac{\ln 2n}{2n} - \sum_{n=1}^{m} \frac{\ln(2n-1)}{2n-1} \\
&= \frac{\ln 2}{2} \sum_{n=1}^{m} \frac{1}{n} + \frac{1}{2} \sum_{n=1}^{m} \frac{\ln n}{n} - \left(\sum_{n=1}^{2m} \frac{\ln n}{n} - \sum_{n=1}^{m} \frac{\ln 2n}{2n} \right) \\
&= \ln 2 H_m + \sum_{n=1}^{m} \frac{\ln n}{n} - \sum_{n=1}^{2m} \frac{\ln n}{n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln(m+n)}{m+n} \\
&= \ln 2 H_m - \sum_{n=1}^{m} \frac{\ln m + \ln(1+n/m)}{m+n} \\
&= \ln 2 H_m - \ln m (H_{2m} - H_m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \\
&= H_m \ln(2m) - H_{2m} \ln m - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \\
&\xlongequal{H_m = \ln m + \gamma + \mathcal{O}(1/m)} \gamma \ln 2 + \mathcal{O}(1/m) - \frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} \qquad (3)
\end{aligned}
$$

Now with (2) and (3), (1) will give

$$
f_m(-1) \to \gamma \ln 2 - \int_0^1 \frac{\ln(1+x)}{1+x}\, dx = \gamma \ln 2 - \frac{\ln^2 2}{2}.
$$

Comment : In fact, one can easily show that

$$
\frac{1}{m} \sum_{n=1}^{m} \frac{\ln(1+n/m)}{1+n/m} = \frac{\ln^2 2}{2} + \mathcal{O}(1/m), \quad \text{so}
$$

$$
\sum_{n=1}^{m} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n} - \ln 2 \right) \cdot \ln\left(\frac{n+1}{n} \right) = \gamma \ln 2 - \frac{\ln^2 2}{2} + \mathcal{O}\left(m^{-1} \ln m \right).
$$

A solution to the problem 5215 of May's 2012 issue of School Science and Mathematics journal

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April 28, 2012

The Problem : Proposed by Neculai Stanciu, Buzău, Romania

Evaluate the integral $\int_{-1}^{1} \frac{2x^{1004}+x^{3014}+x^{2008}\sin x^{2007}}{1+x^{2010}}\,dx.$

Solution : Since $\frac{2x^{1004}+x^{3014}}{1+x^{2010}}$ is even and $\frac{x^{2008}\sin x^{2007}}{1+x^{2010}}$ is odd, we have

$$\begin{aligned}
\int_{-1}^{1} \frac{2x^{1004}+x^{3014}+x^{2008}\sin x^{2007}}{1+x^{2010}}\,dx &= \int_{-1}^{1} \frac{2x^{1004}+x^{3014}}{1+x^{2010}}\,dx + \int_{-1}^{1} \frac{x^{2008}\sin x^{2007}}{1+x^{2010}}\,dx \\
&= 2\int_{0}^{1} \frac{2x^{1004}+x^{3014}}{1+x^{2010}}\,dx + 0 \\
&= 2\int_{0}^{1} x^{1004} + \frac{x^{1004}}{1+x^{2010}}\,dx \\
&= \frac{2}{1005} + 2\int_{0}^{1} \frac{x^{1004}}{1+(x^{1005})^2}\,dx \\
&\overset{x^{1005}=y}{=\!=\!=} \frac{2}{1005} + \frac{2}{1005}\int_{0}^{1} \frac{1}{1+y^2}\,dy \\
&= \frac{2}{1005}\left(1+\frac{\pi}{4}\right).
\end{aligned}$$

A solution to the problem 5217 of May's 2012 issue of School Science and Mathematics journal

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April 25, 2012

The Problem : Proposed by Ovidiu Furdui, Cluj-Napoca, Romania

Find the value of: $\lim_{n\to+\infty}\int_0^1\int_0^1\sqrt[n]{(x^n+y^n)^k}$ *where* k *is a positive real number.*

Solution : It is easily shown that $\sqrt[n]{(x^n+y^n)^k}\to\begin{cases}x^k & ,y\le x\\ y^k & ,x<y\end{cases}$ and since $0\le\sqrt[n]{(x^n+y^n)^k}\le 2^k$, by the dominated convergence theorem we have

$$\begin{aligned}\lim_{n\to+\infty}\int_0^1\int_0^1\sqrt[n]{(x^n+y^n)^k} &= \int_0^1\int_0^1\lim_{n\to+\infty}\sqrt[n]{(x^n+y^n)^k}\\ &= \int_0^1\int_0^x x^k\,dy\,dx+\int_0^1\int_x^1 y^k\,dy\,dx\\ &= \int_0^1 x^{k+1}\,dx+\int_0^1\frac{1-x^{k+1}}{k+1}\,dx\\ &= \frac{2}{k+2}.\end{aligned}$$

A solution to the problem 5181 of November's 2011 issue of School Science and Mathematics journal

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Saturday 12/11/2011

• **The Problem :** Proposed by Proposed by Ovidiu Furdui, Cluj, Romania.

Calculate $\displaystyle\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{nm}{(n+m)!}$.

• **Solution :** The summands being all positive we can sum by triangles :

$$\begin{aligned}
\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{nm}{(n+m)!} &= \sum_{k,\ell,n\in\mathbb{N}\,\wedge\, k+\ell=n}\frac{nm}{(n+m)!}=\sum_{n=2}^{+\infty}\frac{\sum_{\ell=1}^{n-1}(n-\ell)\ell}{n!}\\
&= \frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n-1)n(n+1)}{n!}=\frac{1}{6}\sum_{n=2}^{+\infty}\frac{(n+1)}{(n-2)!}\\
&= \frac{1}{6}\sum_{n=0}^{+\infty}\frac{(n+3)}{n!}=\frac{1}{6}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{dx^{n+3}}{dx}\Big|_{x=1}\\
&= \frac{1}{6}\frac{d}{dx}\left(\sum_{n=0}^{+\infty}\frac{x^{n+3}}{n!}\right)\Big|_{x=1}=\frac{1}{6}\frac{d\left(x^3e^x\right)}{dx}\Big|_{x=1}\\
&= \frac{2e}{3}.
\end{aligned}$$

A solution to the problem 5224 of November's 2012 issue of School Science and Mathematics journal

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January 29, 2013

The Problem : Proposed by Kenneth Korbin, New York, NY.

Let $T_1 = T_2 = 1, T_3 = 2$ *and* $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. *Find the value of*

$$\sum_{n\geq 1} \frac{T_n}{\pi^n}.$$

Solution : We compute the generating function, $f(z) := \sum_{n\geq 1} T_n z^n$, of T_n.

The recurrence is equivalent to

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad n \geq 1,\ T_1 = T_2 = 1, T_3 = 2,$$

so multiplying with z^n and summing for $n \geq 1$ we get

$$\begin{aligned}
&\sum_{n\geq 1} T_{n+3} z^n = \sum_{n\geq 1} T_{n+2} z^n + \sum_{n\geq 1} T_{n+1} z^n + \sum_{n\geq 1} T_n z^n \Leftrightarrow \\
\Leftrightarrow &\frac{1}{z^3}\left(f(z) - z - z^2 - 2z^3\right) = \frac{1}{z^2}\left(f(z) - z - z^2\right) + \frac{1}{z}\left(f(z) - z\right) + f(z) \\
\Leftrightarrow &f(z) = -\frac{z}{z^3 + z^2 + z - 1}.
\end{aligned}$$

Now $g(x) := x^3 + x^2 + x - 1$ is strictly increasing and $g(1/2) < 0$. Since $g(x) \overset{x\to+\infty}{\longrightarrow} +\infty$, $g(x)$ has a single real root, denote it by ρ, with $\rho > 1/2$ and two conjugate complex roots, denote them by $z, \bar{z}$.

By Vieta's relations we get

$$\begin{cases} 2\mathrm{Re}(z) + \rho = -1 \\ 2\rho\mathrm{Re}(z) + |z|^2 = 1 \end{cases} \Rightarrow |z|^2 = \rho^2 + \rho + 1 > \frac{1}{4} \Rightarrow |z| > \frac{1}{2}.$$

The above show that $f(z)$ has radius of convergence $> \dfrac{1}{2}$, so since $1/\pi < 1/2$ we get that

$$\sum_{n\geq 1} \frac{T_n}{\pi^n} = f\left(\frac{1}{\pi}\right) = \frac{\pi^2}{\pi^3 - \pi^2 - \pi - 1}.$$

A solution to the problem 5226 of November's 2012 issue of School Science and Mathematics journal

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October 31, 2012

The Problem : Proposed by D. M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" Secondary School, Buzău, Romania.

Calculate:

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx.$$

where $0<a<b$ *and* $n>0$.

Solution : We have

$$\begin{aligned}
&\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,dx\\
&\overset{x=y+\frac{a+b}{2}}{=\!=\!=\!=} \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{\sqrt[n]{y+\frac{b-a}{2}}(1+\sqrt[n]{\frac{b-a}{2}-y})}{\sqrt[n]{y+\frac{b-a}{2}}+2\sqrt[n]{\left(y+\frac{b-a}{2}\right)\left(\frac{b-a}{2}-y\right)}+\sqrt[n]{\frac{b-a}{2}-y}}-\frac{1}{2}+\frac{1}{2}\,dy\\
&:=\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)+\frac{1}{2}\,dy\\
&=\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} g(y)\,dy+\frac{b-a}{2}.
\end{aligned}$$

Now it is easy to see that $g(y)$ is odd so the given integral equals $\dfrac{b-a}{2}$.

A solution to the problem 5227 of November's 2012 issue of School Science and Mathematics journal

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November 1, 2012

The Problem : Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

Compute:

$$\lim_{n\to+\infty}\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right).$$

Solution : We have

$$\begin{aligned}
\prod_{k=1}^{n}\left(\frac{(n+1)+\sqrt{nk}}{n+\sqrt{nk}}\right) &= \exp\left(\sum_{k=1}^{n}\ln\frac{n+1+\sqrt{nk}}{n+\sqrt{nk}}\right)\\
&= \exp\left(\sum_{k=1}^{n}\ln\left(1+\frac{1}{n+\sqrt{nk}}\right)\right)\\
&= \exp\left(\sum_{k=1}^{n}\frac{1}{n+\sqrt{nk}}+\mathcal{O}(n^{-1})\right)\\
&= e^{\mathcal{O}(n^{-1})}\exp\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\sqrt{k/n}}\right)\\
&\to \exp\left(\int_0^1\frac{1}{1+\sqrt{x}}\,dx\right)\overset{\sqrt{x}=y}{=}\frac{e^2}{4}.
\end{aligned}$$

A solution to the problem 5219 of October's 2012 issue of School Science and Mathematics journal

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November 8, 2012

The Problem : Proposed by David Manes and Albert Stadler, SUNY College at Oneonta, Oneonta, NY and Herrliberg, Switzerland (respectively).

Let k *and* n *be natural numbers. Prove that:*

$$\sum_{j=1}^{n}\cos^k\frac{(2j-1)\pi}{2n+1}=\begin{cases}\dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2}-\dfrac{1}{2} & ,k\,\text{even}\\ \dfrac{1}{2} & ,k\,\text{odd}\end{cases}.$$

Solution : At first, noting that $\cos(\pi-x)=\cos x$, we have that

$$\begin{aligned}\sum_{j=1}^{n}\cos^k\frac{(2j-1)\pi}{2n+1}&=\frac{1}{2}\left(\sum_{j=1}^{2n-1}\cos^k\frac{j\pi}{2n+1}+\sum_{j=1}^{2n-1}(-1)^{j+1}\cos^k\frac{j\pi}{2n+1}\right)\\&=\begin{cases}\displaystyle\sum_{j=1}^{n}\cos^k\frac{j\pi}{2n+1} & ,k\,\text{even}\\ \displaystyle\sum_{j=1}^{n}(-1)^{j+1}\cos^k\frac{j\pi}{2n+1} & ,k\,\text{odd}\end{cases}.\end{aligned}$$

It is easy to see that for a powerseries $f(x)=\sum_{j\geq 0}a_jx^j$, integers $0\leq r<m$ and $w:=e^{2\pi i/m}$, the following identity is true

$$\sum_{j\geq 0}a_{r+jm}x^{r+jm}=\frac{1}{m}\sum_{j=0}^{m-1}w^{-jr}f(w^jx),\qquad |x|<R(=\text{f's radius of convergence}). \tag{1}$$

Now, $[\,\cdot\,]$ denoting the integer part function,

- for k even: setting $k/2 := p,\ 2n+1 := m$, from (1) with $f(x) = (1+x)^{2p}$ and $r = p - m[p/m]$, with $x = 1$ we get

$$\text{LHS} := \sum_{j\geq 0}\binom{2p}{p-m[p/m]+jm} = \frac{1}{m}\sum_{j=0}^{m-1} w^{-j(p-m[p/n])}(1+w^j)^{2p} := \text{RHS}$$

But

$$\text{LHS} = \sum_{j=0}^{2[p/m]}\binom{2p}{p-m[p/m]+jm} = \sum_{j=-[p/m]}^{[p/m]}\binom{2p}{p+jm}$$

and

$$\begin{aligned}\text{RHS} &= \frac{1}{m}\sum_{j=0}^{m-1} e^{-2rji\pi/m}\left(1+e^{i2j\pi/m}\right)^{2p} = \frac{2^{2p}}{m}\sum_{j=0}^{m-1}\cos^{2p}\frac{j\pi}{m}e^{i2j(p-r)\pi/m}\\ &= \frac{2^{2p}}{m}\sum_{j=0}^{m-1}\cos^{2p}\frac{j\pi}{m}e^{2ij[p/m]\pi} = \frac{2^{2p}}{m}\sum_{j=0}^{m-1}\cos^{2p}\frac{j\pi}{m}\end{aligned}$$

so using again $\cos(\pi - x) = \cos x$ we get

$$\sum_{j=1}^{(m-1)/2}\cos^{2p}\frac{j\pi}{m} = \frac{m}{2^{2p+1}}\sum_{j=-[p/m]}^{[p/m]}\binom{2p}{p+jm} - \frac{1}{2},$$

or with the initial notation

$$\sum_{j=1}^{n}\cos^{k}\frac{j\pi}{2n+1} = \frac{2n+1}{2^{k+1}}\sum_{j=-[(k/2)/(2n+1)]}^{[(k/2)/(2n+1)]}\binom{k}{k/2+j(2n+1)} - \frac{1}{2}.$$

- For k odd: setting $2n+1 := m$, from (1) with $f(x) = (1+x)^k$ and $r = \dfrac{k+m}{2} - m\left[\dfrac{k+m}{2m}\right]$, with $x = 1$ and following the same procedure we get

$$\sum_{j=1}^{n}(-1)^{j+1}\cos^{k}\frac{j\pi}{2n+1} = \frac{1}{2} - \frac{2n+1}{2^{k+1}}\sum_{j=-[(k/2)/(2n+1)]}^{[(k/2)/(2n+1)]}\binom{k}{(k+2n+1)/2+j(2n+1)}.$$

From the above we see that

$$\sum_{j=1}^{n}\cos^{k}\frac{(2j-1)\pi}{2n+1} = \begin{cases}\dfrac{2n+1}{2^{k+1}}\dbinom{k}{k/2} - \dfrac{1}{2} & ,k = \text{even and } k < 4n+2\\[2ex] \dfrac{1}{2} & ,k = \text{odd and } k < 2n+1\end{cases}$$

Comment : Note that the above restrictions for k are necessary, since for example it is

$$\sum_{j=1}^{1} \cos^6 \frac{(2j-1)\pi}{2\cdot 1+1} = \frac{1}{64} \neq -\frac{1}{32} = \frac{2\cdot 1+1}{2^{6+1}}\binom{6}{6/2} - \frac{1}{2}$$

and

$$\sum_{j=1}^{1} \cos^3 \frac{(2j-1)\pi}{2\cdot 1+1} = \frac{1}{8} \neq \frac{1}{2}.$$

The above approach, along with some other results, is presented in [1].

References

[1] Mircea Merca *A Note on Cosine Power Sums*, Journal of Integer Sequences, Vol. 15 (2012), Article 12.5.3.

A solution to the problem 5222 of October's 2012 issue of School Science and Mathematics journal

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October 30, 2012

The Problem : Proposed by José Luis Díaz-Barrero, Polytechnical University of Catalonia, Barcelona, Spain.

Calculate without the aid of a computer the following sum

$$\sum_{n\geq 0}(-1)^n(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n, \quad \textit{where} \quad i=\sqrt{-1}$$

.

Solution : From

$$\sum_{n\geq 0}\frac{1}{1-z}, \quad |z|<1$$

differentiating and multiplying with z we get

$$\sum_{n\geq 0} nz^n = \frac{z}{(1-z)^2}, \quad |z|<1.$$

Applying the same again we get

$$\sum_{n\geq 0} n^2z^n = \frac{(1+z)z}{(1-z)^3}, \quad |z|<1.$$

The above yield

$$\sum_{n\geq 0}(n^2+4n+3)z^n = \frac{3-z}{(1-z)^3}, \quad |z|<1.$$

Now

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n(n+1)(n+3)\left(\frac{1}{1+2\sqrt{2}i}\right)^n &= \sum_{n\geq 0}(n^2+4n+3)\left(-\frac{1}{1+2\sqrt{2}i}\right)^n \\
&= \frac{3+\frac{1}{1+2\sqrt{2}i}}{\left(1+\frac{1}{1+2\sqrt{2}i}\right)^3} \\
&= \frac{(1+2\sqrt{2}i)^2(2+3\sqrt{2}i)}{4(1+\sqrt{2}i)^3} \\
&= \frac{41}{27}+\frac{103\sqrt{2}}{108}i.
\end{aligned}$$

A solution to the problem 5223 of October's 2012 issue of School Science and Mathematics journal

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October 31, 2012

The Problem : Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.

a) *Find the value of*

$$\sum_{n\geq 0}(-1)^n\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right).$$

b) *More generally, if* $x\in(-1,1]$ *is a real number, calculate*

$$\sum_{n\geq 0}(-1)^n\left(\frac{x^{n+1}}{n+1}-\frac{x^{n+1}}{n+2}+\frac{x^{n+3}}{n+3}-\cdots\right).$$

Solution : We answer to b). From this, we immediately get 1/2 as the answer for a).
We have

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n\sum_{k\geq 1}(-1)^{k-1}\frac{x^{n+k}}{n+k} &= \sum_{n\geq 0}\sum_{k\geq 1}(-1)^{n+k-1}\frac{x^{n+k}}{n+k}\\
&\overset{m:=n+k}{=\!=\!=} \sum_{n\geq 0}\sum_{m\geq n+1}(-1)^{m-1}\frac{x^m}{m}\\
&\overset{x\in(-1,1]}{=\!=\!=} \sum_{n\geq 0}\left(\ln(1+x)-\sum_{m=1}^{n}(-1)^{m-1}\frac{x^m}{m}\right)\\
&= \lim_{N\to+\infty}\sum_{n=0}^{N}\left(\ln(1+x)-\sum_{m=1}^{n}(-1)^{m-1}\frac{x^m}{m}\right)\\
&:= \lim_{N\to+\infty}A_N(x).
\end{aligned}$$

Now for $x \in (-1,1]$ it is

$$A_N'(x) = \sum_{n=0}^{N}\left(\frac{1}{1+x} - \sum_{m=1}^{n}(-x)^{m-1}\right) = \sum_{n=0}^{N}\frac{(-x)^n}{1+x} = \frac{1-(-x)^{N+1}}{(1+x)^2}$$

so integrating we get

$$A_N(x) = \int_0^x \frac{1-(-y)^{N+1}}{(1+y)^2}\,dy = \frac{x}{1+x} + (-1)^N\int_0^x \frac{y^{N+1}}{(1+y)^2}\,dy.$$

But for $x \in (-1,1)$ it is

$$\Big|\int_0^x \frac{y^{N+1}}{(1+y)^2}\,dy\Big| \leq \int_0^{|x|}\frac{y^{N+1}}{(1-y)^2}\,dy \leq |x|^{N+1}\int_0^{|x|}\frac{1}{(1-y)^2}\,dy = \frac{|x|^{N+2}}{1-|x|} \to 0$$

and for $x=1$, $\displaystyle\int_0^1 \frac{y^{N+1}}{(1+y)^2}\,dy \to 0$ by the Dominated Convergence theorem, so

$$\sum_{n\geq 0}(-1)^n\sum_{k\geq 1}(-1)^{k-1}\frac{x^{n+k}}{n+k} = \frac{x}{x+1}, \qquad x \in (-1,1].$$

A solution to the problem 5174 of October's 2011 issue of School Science and Mathematics journal

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Saturday 12/11/2011

• **The Problem :** Proposed by Josè Luis Diaz – Barrero, Barcelona, Spain.

Let n be a positive integer, Compute $\lim_{n\to+\infty}\frac{n^2}{2^n}\sum_{k=0}^{n}\frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}$.

• **Solution :** For $n\in\mathbb{N}$ and and $x\in\mathbb{R}$ we got $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$ so $x^4(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^{k+4}$. Now differentiate to get

$$4x^3(1+x)^n+nx^4(1+x)^{n-1}=\sum_{k=0}^{n}(k+4)\binom{n}{k}x^{k+3},\qquad \text{so}$$

$$4(1+x)^n+nx(1+x)^{n-1}=\sum_{k=0}^{n}(k+4)\binom{n}{k}x^k.$$

Now integrate on $[0,x]$ to get

$$\frac{3(1+x)^{n+1}}{n+1}+x(1+x)^n-\frac{3}{n+1}=\sum_{k=0}^{n}\frac{(k+4)}{k+1}\binom{n}{k}x^{k+1}, \text{ once more to get}$$

$$\frac{2(1+x)^{n+2}}{(n+1)(n+2)}+\frac{x(1+x)^{n+1}}{n+1}-\frac{3x}{n+1}-\frac{2}{(n+1)(n+2)}=\sum_{k=0}^{n}\frac{(k+4)}{(k+1)(k+2)}\binom{n}{k}x^{k+2}$$

and again to get

$$\sum_{k=0}^{n}\frac{(k+4)}{(k+1)(k+2)(k+3)}\binom{n}{k}x^{k+3}=$$

$$\frac{(1+x)^{n+3}}{(n+1)(n+2)(n+3)}+\frac{x(1+x)^{n+2}}{(n+1)(n+2)}-\frac{3x^2}{2(n+1)}-\frac{2x}{(n+1)(n+2)}-\frac{1}{(n+1)(n+2)(n+3)}.$$

Setting $x=1$ above,we easily see that

$$\frac{n^2}{2^n}\sum_{k=0}^{n}\frac{(k+4)}{(k+1)(k+2)(k+3)}\binom{n}{k}\overset{n\to+\infty}{\longrightarrow}4.$$

A solution to the problem 5175 of October's 2011 issue of School Science and Mathematics journal

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Saturday 12/11/2011

• **The Problem :** Proposed by Proposed by Ovidiu Furdui, Cluj, Romania.

Find the value of $\lim_{n\to+\infty}\frac{1}{n}\sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}$.

• **Solution :** Setting $a_n := \sum_{i,j=1}^{n}\frac{i+j}{i^2+j^2}$ we will show that $a_{n+1}-a_n \to \frac{\pi}{2}+\ln 2$, so by the $\mathrm{Cesàro\,Stolz}$ theorem the desired limit will also be equal to $\frac{\pi}{2}+\ln 2$.

We have

$$\begin{aligned}
a_{n+1}-a_n &= \sum_{i=1}^{n+1}\sum_{j=1}^{n+1}\frac{i+j}{i^2+j^2}-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}\\
&= 2\sum_{k=1}^{n}\frac{k+n+1}{k^2+(n+1)^2}+\frac{1}{n+1}\\
&= \frac{2}{n}\sum_{k=1}^{n}\frac{k/n+1+1/n}{(k/n)^2+(1+1/n)^2}+\frac{1}{n+1}\\
&= \frac{2}{n}\sum_{k=1}^{n}\frac{k/n+1+1/n}{(k/n)^2+1+\mathcal{O}(n^{-1})}+\frac{1}{n+1}\\
&= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{k/n+1+1/n}{(k/n)^2+1}\big(1+\mathcal{O}(n^{-1})\big)^{-1}\right)+\frac{1}{n+1}\\
&= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{k/n+1+1/n}{(k/n)^2+1}\big(1+\mathcal{O}(n^{-1})\big)\right)+\frac{1}{n+1}\\
&= \frac{(2+\mathcal{O}(n^{-1}))}{n}\sum_{k=1}^{n}\frac{k/n+1}{(k/n)^2+1}+\sum_{k=1}^{n}\big(\mathcal{O}(n^{-2})+\mathcal{O}(n^{-3})\big)+\mathcal{O}(n^{-1})\\
&\to 2\int_0^1\frac{1+x}{1+x^2}=\frac{\pi}{2}+\ln 2.
\end{aligned}$$