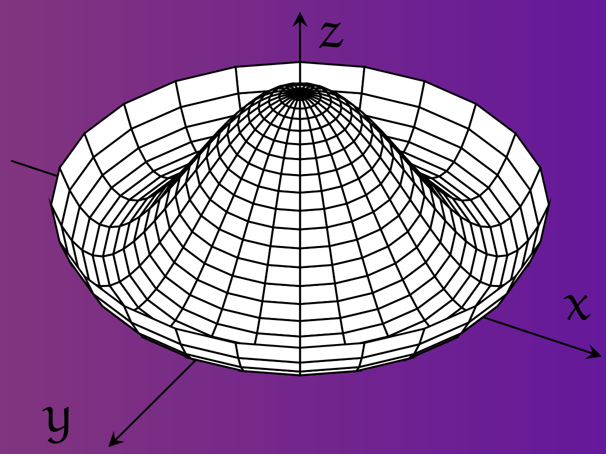
Mathematical Analysis

A collection of problems

Real & Complex Analysis • General Topology • Multivariable Calculus • Integrals and Series



Tolaso J. Kos

Mathematical Analysis

A collection of problems

TOLASO NETWORK

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Mini Contents

Foreword

Dear reader,

there are a lot of interesting analysis problems scattered in the Internet World. Navigating through different sites you may encounter an exercise that will catch your attention and possibly you may want to archive it in your collection so to have access to it later. This is the main idea behind this booklet. The attempt started back in 2014 when an effort to collect as many exercises as possible began. Basic ideas are being recycled frequently and reappear in many exercises although unrelated at first.

The booklet contains a collection of interesting problems in Mathematical Analysis. The problems come from various branches of mathematics.

- Real and Complex Analysis
- General Topology
- Multivariable Calculus
- Integrals and Series

In each section the reader of this booklet shall encounter exercises that may find out there. Many of them are known to you but still they are interesting. However , there do exist exercises that demand creativity in order to be solved. The level of difficulty varies from exercise to exercise and in no way are the problems ordered according to their level of difficulty.

The author (Tolaso) started the collection of the problems using exercises that he encountered in his university classes (Calculus I, Calculus II and Calculus IV) and found to be the most interesting and fascinating. He dediced to include non trivial problems (as these have nothing to offer usually and rely mostly on definitions) but challenging ones.

The version you are now reading is Version 9 which is an improvement of the previous Version 8. I would like to personally thank all those people who contacted me personally to mention any typographical errors and / or mathematical errors that were corrected in this version. A big thanks to all of you guys ! I am open to your e-mails for improvements / suggestions . Feel free to contact me at the e-mail address that you will find in page 2. Last but not least , you are free to use the booklet as an instructive tutorial to your students. However , be very careful when assigning exercises to them.

Tolaso J Kos

May 18, 2018

Acknowledgements

- Many thanks to all those people (from all around the world) who embraced this booklet and have sent remarks and / or suggestions so that it is improved as well as selecting some of its exercises to assign to their students. I really appreciate it.
- The people at TeX Stack Exchange who have suggested some hacks for some parts of the existing code so that everything fits within the specified margins as well as the suggestion for the first page.

Donation

If you like the work done for this booklet as well as the overall work produced by Tolaso Network and want to donate please follow the link found at page 2. We thank you in advance.

Real - Complex Analysis

1. For which $a \in \mathbb{R}$ does the sequence
$$\gamma_n = (1+a)(1+2a^2)\cdots(1+na^n)$$
converge? Give a brief explanation.

2. We define a sequence x_n as follows
$$x_{n+3} = \frac{x_{n+2}^2 + 5x_{n+1}^2 + x_n^2}{x_{n+2} + 5x_{n+1} + x_n}$$
where $x_1, x_2, x_2 > 0$. Examine whether the sequence converges.

3. A sequence of real number $\{x_n\}_{n\in\mathbb{N}}$ satisfies the condition
$$|x_n - x_m| > \frac{1}{n} \quad \text{whenever} \quad n < m$$
Prove that x_n is not bounded.

4. Prove that
$$\lim_{n\to+\infty} \left((n+1)^{(n+2)/(n+1)} - n^{(n+1)/n} \right) = 1$$

5. Prove that
$$\lim_{n\to+\infty} n \sin(2\pi e n!) = 2\pi$$

6. Prove that the limit
$$\ell = \lim_{n\to+\infty} \frac{\tan n}{n}$$
does not exist.

7. Find the value of
$$\ell = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{\cdots}}}}$$

8. Let $\lfloor \cdot \rfloor$ denote the floor function. Define
$$a_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor$$
   (a) Prove that the limit points of a_n is the set $[0,1]$.
   (b) Prove that $\limsup a_n = 1$.

9. Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and $\{y_n\}_{n=1}^{\infty} \subset (0,+\infty)$. Suppose that $\{x_n/y_n\}_{n=1}^{\infty}$ is monotone. Prove that the sequence $\{z_n\}_{n\in\mathbb{N}}$ defined as
$$z_n = \frac{x_1 + x_2 + \cdots + x_n}{y_1 + y_2 + \cdots + y_n}$$
is also monotone.

10. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined as
$$x_n = \sin 1 + \sin 3 + \sin 5 + \cdots + \sin(2n-1)$$
Find the supremum as well as the infimum of the sequence x_n.

11. Let $\alpha \in \mathbb{R}$ such that $\alpha/\pi \notin \mathbb{Q}$. Prove that the sequence
$$\omega_n = \sin(\sin\alpha) + \sin(\sin(2\alpha)) + \cdots + \sin(\sin(n\alpha))$$
is bounded.

12. Define
$$f_n(x) = \frac{x^n}{n!} \quad , \quad x \in \mathbb{R} \, , \; n \in \mathbb{N}$$
Examine the pointwise convergence as well as the uniform convergence of f_n.

13. Given the sequence of functions
$$f_n(x) = \cos^n x, \; 0 \leqslant x \leqslant \pi$$
Prove that
   (a) $\lim f_n(x) = 0$ but $f_n(\pi)$ does not converge.
   (b) Prove that f_n converges pointwise but not uniformly on $[0, \pi/2]$.

14. Let $\{a_n\}_{n\in\mathbb{N}}$ be a real valued sequence such that the series $\sum\limits_{n=1}^{\infty} a_n^2$ converges. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{n}$ also converges.

15. Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive real valued sequence. If the series $\sum\limits_{n=1}^{\infty} a_n$ converges prove that the series $\sum\limits_{n=1}^{\infty} a_n^{n/(n+1)}$ also converges.

16 Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and let us denote with $\lfloor \cdot \rfloor$ the floor function. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\alpha - \frac{\lfloor n\alpha \rfloor}{n} \right)$$

diverges.

(16th Cuban Mathematical Olympiad)

17 Let a_n be a positive and strictly decreasing sequence such that $\lim a_n = 0$. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{a_n - a_{n+1}}{a_n}$$

diverges.

18 Let $\mathbb{P}$ denote the set of prime numnbers. Discuss the convergence of the series

$$\mathcal{S} = \sum_{p \in \mathbb{P}} \frac{\sin p}{p}$$

19 Examine whether the (double) series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin(\sin(nm))}{n^2 + m^2}$$

converges.

20 Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of strictly increasing positive integers. For each $n \geqslant 1$ let W_n be the least common multiple of the first n terms $X_1, X_2, \ldots, X_n$. Prove that , as $n \to +\infty$, the series

$$\mathcal{S} = \frac{1}{W_1} + \frac{1}{W_2} + \cdots + \frac{1}{W_n}$$

converges.

21 Let $\{a_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. Prove that the series $\displaystyle\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]}$ converges. Here $[\cdot, \cdot]$ denotes the least common multiple.

22 Let $f : (0, +\infty) \to \mathbb{R}$ be a positive differentiable function such that its derivative is positive. Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges if-f the series $\displaystyle\sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^2}$ converges.

23 Let $\mathcal{H}_n$ denote the n-th harmonic number. Study the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \alpha^{\mathcal{H}_n}$$

for the different values of $\alpha > 0$.

24 Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that the series

$$\mathcal{S} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt[n]{\log n!}}{\log\left(\mathcal{H}_{n+1}\right)}$$

converges.

25 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\log(\mathcal{H}_n)}{e^{\mathcal{H}_n}}$$

converges.

Hint: Let $x_1, \ldots, x_n \in (0, 1)$. It holds that

$$\sum_{i=1}^{n} (1 - x_i) \geqslant 1 - \prod_{i=1}^{n} x_i$$

It appears that this problem is quite difficult. It appeared in several fora including math.stackexchange.com as well as mathematica.gr. In both went answered till today. In math.stackexchange.com they suggest that the series converges and its limit is $\frac{1}{2}$.

Hint:

$$\begin{aligned}
\sum_{i=0}^{n} \frac{1}{[a_i, a_{i+1}]} &= \sum_{i=0}^{n} \frac{(a_i, a_{i+1})}{a_i a_{i+1}} \\
&\leqslant \sum_{i=0}^{n} \frac{a_{i+1} - a_i}{a_i a_{i+1}} \\
&= \sum_{i=0}^{n} \frac{1}{a_i} - \frac{1}{a_{i+1}} \\
&= \frac{1}{a_0} - \frac{1}{a_n} < \frac{1}{a_0}
\end{aligned}$$

26 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n^{\mathcal{H}_n}}{(\mathcal{H}_n)^n}$$

converges.

27 Let $\{a_n\}_{n\in\mathbb{N}}$ be a positive real valued sequence such that the series $\sum\limits_{n=1}^{\infty} a_n$ converges. Examine the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(1 - \frac{\sin a_n}{a_n}\right)$$

28 Let $\{x_n\}_{n\in\mathbb{N}}$ be a real valued sequence of positive terms such that $\sum\limits_{n=1}^{\infty} x_n$ converges. Set

$$s_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$$

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{x_n s_n^2}$ converges.

29 Let $\alpha \in \mathbb{R}$. For which values of α does the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arcsin \frac{n}{n+4}\right)^{\alpha}$$

converge?

30 Examine the convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\sin(\sin n)}{n}$$

Does it converge absolutely? Justify your answer.

31 Let a_n be a sequence of positive terms and suppose that $\sum\limits_{n=1}^{\infty} a_n$ converges.

(a) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sum_{k=1}^{n} a_k}$ also converges.

(b) Find the smallest possible value of λ such that

$$\sum_{n=1}^{\infty} \frac{n}{\sum_{k=1}^{n} a_k} \leqslant \lambda \sum_{n=1}^{\infty} \frac{1}{a_n}$$

32 Prove that the series

$$\mathcal{S}_\alpha = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha} \sin(\log n)$$

converges if and only if $\alpha > 0$.

33 For what values of $x \in \mathbb{R}$ do the series

$$\text{(i)}\ \ \mathcal{S}_1 = \sum_{n=1}^{\infty} \cos(2^n x) \qquad \text{(ii)}\ \ \mathcal{S}_2 = \sum_{n=1}^{\infty} \sin(2^n x)$$

converge?

34 Define x_n recursively as:

$$x_1 = 1 \quad , \quad x_{n+1} = \sin x_n$$

(a) Prove that $x_n \sim \sqrt{\frac{3}{n}}$.

(b) Prove that x_n converges to 0 monotonically decreasing.

(c) What inequality should β satisfy in order the series

$$\mathcal{S} = \sum_{n=1}^{\infty} x_n^{\beta}$$

to converge?

35 What can you say about the uniform convergence of the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi x)\,, \quad x \in \mathbb{R}$$

36 Let $x \in \mathbb{R}$. Consider the series

$$\mathcal{S} = \sum_{n=2}^{\infty} \frac{\sin nx}{\log n} \tag{1}$$

(A) (a) Prove that $\mathcal{S}$ converges forall $x \in \mathbb{R}$.

Hint: It holds that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin n\pi x = \begin{cases} \dfrac{\pi x}{2} & , \quad 0 \leqslant x < 1 \\ 0 & , \quad x = 1 \\ \dfrac{\pi(x-2)}{2} & , \quad 1 < x \leqslant 2 \end{cases}$$

(b) Prove that (1) is not a Fourier series of a Lebesgue integrable function.

(B) Examine if the function defined at (1) is continuous. Give a brief explanation to support your argument.

(C) Prove that the series $\sum\limits_{n=2}^{\infty} \frac{\cos nx}{\log n}$ is both Riemann and Lebesgue integrable as well as a Fourier series.

37 Let $a \in \mathbb{Z}$. Define the function

$$f(x) = \sin ax, \;\; x \in (0, \pi)$$

Prove that f can be expanded into a Fourier cosine series and that it holds

$$\sin ax \sim \begin{cases} \dfrac{4a}{\pi} \displaystyle\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{a^2-(2n+1)^2} & , \;\; a \text{ even} \\ \dfrac{4a}{\pi}\left[\dfrac{1}{2a^2} + \displaystyle\sum_{n=1}^{\infty} \frac{\cos 2nx}{a^2-4n^2}\right] & , \;\; a \text{ odd} \end{cases}$$

38 Let $\{a_n\}_{n \in \mathbb{N}}$ be a bounded sequence. Prove that the sequence of functions defined as $\sum\limits_{n=1}^{\infty} \frac{a_n}{n^{2x}}$ converges absolutely and uniformly on $(0, +\infty)$ to a differentiable function.

(Question from a Real Analysis Exam
University of Ioannina , Greece)

39 Let $f : [-\pi, \pi] \rightarrow \mathbb{R}$ be defined as $f(x) = |x|$.

(a) Expand f in a Fourier series.

(b) Prove that

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(c) Apply Parseval's identity to evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Do the same question for the quite similar series $\sum\limits_{n=2}^{\infty} \frac{\sin nx}{n \log n}$.

40 Examine if there exists an $1-1$ function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \frac{f(n)}{n^2}$ converges.

41 Examine whether the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \sin\left[\pi\left(2+\sqrt{3}\right)^n\right]$$

converges.

42 Examine whether the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(e - \left(1+\tfrac{1}{n}\right)^n\right)$$

converges.

43 Let $\{a_n\}_{n \in \mathbb{N}}$ be a real valued sequence such that the series $\sum\limits_{n=1}^{\infty} \frac{a_n}{n}$ converges. Prove that

$$\lim_{n \rightarrow +\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$$

44 Given the sequence of $f_n : \mathbb{R} \rightarrow \mathbb{R}$ where $n \in \mathbb{N}$ defined as

$$f_n(x) = \sum_{n=1}^{\infty} \frac{n}{n^3 + x^2}$$

prove that

(a) the serieses $\sum\limits_{n=1}^{\infty} f_n$ and $\sum\limits_{n=1}^{\infty} f'_n$ converge uniformly to functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$.

(b) the functions f, g are continuous.

(c) $f' = g$.

(d) it holds that

(i) $\displaystyle\int_{-1}^{1} f(x)\, dx = 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \arctan \frac{1}{n\sqrt{n}}$

(ii) $\displaystyle\int_{-\pi}^{\pi} x^4 g(x)\, dx = 0.$

45 Consider the real valued sequence $\{y_n\}_{n \in \mathbb{N}}$ such that forall real valued sequences $\{x_n\}_{n \in \mathbb{N}}$ with $\lim x_n = 0$ the series $\sum\limits_{n=1}^{\infty} x_n y_n$ converges. Prove that the series $\sum\limits_{n=1}^{\infty} |y_n|$ also converges.

What can you say about the integral $\int_{-\infty}^{\infty} f(t)\, dt$? Does it converge?

46 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly throughout $\mathbb{R}$ if and only if $na_n \to 0$.

47 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive terms. Prove that the series $\sum\limits_{n=1}^{\infty} a_n \cos nx$ converges uniformly on $\mathbb{R}$ if and only if the series $\sum\limits_{n=1}^{\infty} a_n$ converges.

48 Let $\mathcal{H}_n$ denote the n -th Harmonic number. Prove the inequality

$$\frac{\pi^2}{6}\left(\zeta(3)-\frac{\pi^2}{12}\right) < \sum_{n=1}^{\infty} \frac{e^{\mathcal{H}_n}\log \mathcal{H}_n}{n^3}$$

49 Let $f:\mathbb{R}\to\mathbb{R}$ be an integrable function. Prove that

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}f\left(x-\sqrt{n}\right)$$

converges for almost all x.

50 Prove that the series

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{\cos(\log k)}{k}$$

diverges by first proving that

$$\sum_{n=1}^{N}\frac{\cos\log n}{n}=\sin\log N+\mathfrak{Re}\,\zeta(1+i)+\mathcal{O}(N^{-1})$$

51 What is the monotony of the function

$$f(j)=\prod_{i=-j}^{0}\sum_{k=0}^{\infty}\frac{i^k}{k!},\quad j\in\mathbb{Z}$$

52 Let $f:[-\pi,\pi]\to\mathbb{R}$ be a Riemann integrable function. Prove that

$$\lim_{n\to+\infty}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\lim_{n\to+\infty}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$$

53 Prove , without using special functions, that the integral $\displaystyle\int_0^{\pi}\frac{\ln x}{x+\pi}\,dx$ converges.

54 Let $f_n(x):[0,1]\to\mathbb{R}$ be a sequence of functions converging uniformly to a function f. Prove that

$$\lim_{n\to+\infty}\int_{1/n}^{1}f_n(x)\,dx=\int_0^1 f(x)\,dx$$

55 Let $f,g:\mathbb{R}\to\mathbb{R}$ be 1 periodic and continuous functions. Prove that

$$\lim_{n\to+\infty}\int_0^1 f(x)g(nx)\,dx=\int_0^1 f(x)\,dx\int_0^1 g(x)\,dx$$

56 Let $f,g:[0,1]\to\mathbb{R}$ be continuous functions such that $0<f(x)<cg(x)$ forall $x\in(0,1)$. for some constant c. Evaluate the limit:

$$\ell=\lim_{n\to+\infty}\int_0^1\cdots\int_0^1\frac{f(x_1)+\cdots+f(x_n)}{g(x_1)+\cdots+g(x_n)}\,d(x_1,\ldots,x_n)$$

57 Evaluate the limit

$$\ell=\lim_{n\to+\infty}\int_0^1\cdots\int_0^1\frac{x_1^2+\cdots+x_n^2}{x_1+\cdots+x_n}\,d(x_1,\ldots,x_n)$$

58 Let $f:(0,+\infty)\to\mathbb{R}$ such that , forall $x>0$, the limit $\lim\limits_{n\to+\infty}f(nx)\in\mathbb{R}$. Examine if the limit $\lim\limits_{x\to+\infty}f(x)$ exists in $\mathbb{R}$ if:

(a) f is a continuous function.

(b) f is an arbitrary function.

59 (a) Give an example of a bounded function $f:(0,+\infty)\to\mathbb{R}$ such that the limit $\ell=\lim\limits_{x\to 0^+}f(x)$ does not exist.

(b) If f is a function such as described in (a) then examine if the following limits exist.

(i)
$$\ell_1=\lim_{x\to 0^+}xf(x)$$

(ii)
$$\ell_2=\lim_{x\to 0^+}(1-x)f(x)$$

You might consider ideas from this link.

60 Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to+\infty}\int_a^b \frac{f(x)}{3+2\cos nx}\,dx=\frac{1}{\sqrt{5}}\int_a^b f(x)\,dx$$

61 Evaluate

$$\ell=\lim_{n\to+\infty} n^{-n^2}\left[\prod_{k=0}^{n-1}\left(n+\frac{1}{2^k}\right)\right]^n$$

62 Prove that

$$\min_{a_i\in\mathbb{R}}\int_0^1\left|x^n+a_1x^{n-1}+\cdots+a_n\right|\,dx=\frac{1}{4^n}$$

63 Let p, q be two points and γ be a curve passing through these two points. Prove that

(a) $\gamma'(t)\cdot u\leqslant\|\gamma'(t)\|$ where u is an arbitrary unit vector.

(b) that the segment of the curve γ between the points p and q has length at least equal to the distance $\|q-p\|$ by considering as $u=\frac{q-p}{\|q-p\|}$.

[1]

64 Let $f:\mathbb{R}^n\to\mathbb{R}$ be a smooth function. Prove that there exist functions $g_i,\ i=1,\dots,n$ such that

$$f(x_1,x_2,\dots,x_n)-f(0,0,\dots,0)=\sum_{i=1}^n x_i g_i(x_1,x_2,\dots,x_n)$$

65 Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function such that $f'(x)=0$ forall $x\in\mathbb{Q}$. Does it necessarily follow that f is constant throughout $\mathbb{R}$? Explain your answer.

66 Find all functions $f:\mathbb{R}\to\mathbb{R}$ that preserve convergent series. *(That is a function preserves convergent series in the sense mentioned above if $\sum f(a_n)$ converges whenever $\sum a_n$ converges.)* [2]

67 Examine if there exists a function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(f(x))=x^2+1 \text{ forall } x\in\mathbb{R}$$

[1] The conclusion of this exercise is to show that the line is the shortest distance between two points.

[2] The answer to this difficult question is that the only functions with this property are of the form $f(x)=\lambda x,\ x\in(-\delta,\delta)$.

68 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that:

$$f(x)=f(x+1)=f(x+2\pi)\quad,\quad\forall x\in\mathbb{R}$$

Prove that f is constant.

69 Given a continuous function $f:\mathbb{R}\to\mathbb{R}$ such that forall $x\in\mathbb{R}\setminus\{0,1\}$

$$\int_0^x f(t)\,dt>\int_x^1 f(t)\,dt \tag{1}$$

prove that $\int_0^1 f(t)\,dt=0$.

70 Let $f:[a,b]\to\mathbb{R}$ be a continuously differentiable function such that $f(a)=f(b)=0$ and $\int_a^b f^2(t)\,dt=1$. Prove that:

(a) $\displaystyle\int_a^b xf(x)f'(x)\,dx=-\frac{1}{2}$

(b) $\displaystyle\int_a^b\left(f'(x)\right)^2\,dx\int_a^b x^2f^2(x)\,dx>\frac{1}{4}$

71 Let

$$f(x)=\sin x\sin(2x)\sin(4x)\cdots\sin(2^nx)$$

Prove that

$$|f(x)|\leqslant\frac{2}{\sqrt{3}}\left|f\left(\frac{\pi}{3}\right)\right|$$

72 Prove that for every $x\in\mathbb{R}$ the inequality

$$\frac{x^{2n}}{(2n)!}+\frac{x^{2n-1}}{(2n-1)!}+\cdots+\frac{x^2}{2!}+x+1>0$$

holds.

73 Prove that for arbitrary real numbers $a_1,a_2,\dots,a_n$ the following inequality holds.

$$\sum_{m,n=1}^k\frac{a_ma_n}{m+n}\geqslant 0$$

[3]

[3] A solution goes along these lines:

$$\sum_{m,n=1}^k\frac{a_ma_n}{m+n}=\sum_{m,n=1}^k\int_0^1 a_ma_nt^{m+n-1}\,dt$$

74 Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n\to+\infty}\left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geqslant e$$

75 Let $\mathcal{C}$ denote the Cantor set. We define the function $\chi_{\mathcal{C}}:[0,1]\to\mathbb{R}$ as follows:

$$\chi_{\mathcal{C}}=\begin{cases}1 & , \quad x\in\mathcal{C}\\ 0 & , \quad \text{elsewhere}\end{cases}$$

(a) Prove that $\chi_{\mathcal{C}}$ is Riemann integrable.

(b) Evaluate $\int_0^1 \chi_{\mathcal{C}}(x)\,dx$.

76 Prove that the function $f:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ defined as

$$f(\mathbf{x})=\frac{\mathbf{x}}{\|\mathbf{x}\|^a}\ ,\ a>0$$

is a vector field but its domain is not star-shaped.

77 Does the ordered field of the rational functions satisfy the axiom of completeness? Explain your answer.

78 Let $f:[2,+\infty)\to\mathbb{R}$ be a uniformly continuous function. Prove that the integral

$$\mathcal{J}=\int_2^\infty \frac{f(x)}{x^2\log^2 x}\,dx$$

converges.

79 Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous and strictly convex function such that $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=+\infty$. Prove that the integral $\int_0^\infty \sin f(x)\,dx$ converges but not absolutely.

$$\begin{aligned}&=\int_0^1\left(\sum_{m,n=1}^k a_m a_n t^{m+n-1}\right)dt\\ &=\int_0^1\left(\sum_{m=1}^k a_m t^{m-1/2}\right)^2 dt\\ &\geqslant 0\end{aligned}$$

In fact the above inequality tells us that the matrix $\left[\frac{1}{m+n}\right]_{m,n=1}^k$ is positive semidefinite.

This is a very difficult exercise. One solution may be found at M. Hata's notes. Another solution is to contradict the result and move along those lines.

I currently have no solution to this , demanding , exercise. It was an exam's question.

80 Let $f:[a,b]\to\mathbb{R}$ be a Riemann integrable function. If $f(x)=0$ forall rationals of the interval $[a,b]$ then prove that $\int_a^b f(x)\,dx=0$.

81 Prove that there exists no rational function such that

$$f(n)=1+\frac{1}{2}+\cdots+\frac{1}{n}$$

forall $n\in\mathbb{N}$.

82 Let $f:\mathbb{R}\to(0,+\infty)$ be a function such that forall $x\in\mathbb{R}$ it holds that

$$f(x)\log f(x)=e^x \tag{1}$$

Evaluate the limit

$$\ell=\lim_{x\to+\infty}\left(1+\frac{\log x}{f(x)}\right)^{f(x)/x}$$

(Romania , 1986)

83 Let $n\in\mathbb{N}$ and let $f:[-1,1]\to\mathbb{R}$ be a continuous function such that

$$\int_{-1}^1 x^{2n}f(x)\,dx=0$$

Prove that f is odd.

84 Let $\mathcal{G}$ denote the Catalan constant. Prove that

$$\log\left(1+\sqrt{2}\right)<\int_0^1\frac{\tanh x}{x}\,dx<\mathcal{G}$$

85 Evaluate the limit

$$\Omega=\lim_{n\to+\infty}\sum_{k=1}^n\frac{\frac{1}{n}\arctan\left(\frac{k}{n}\right)}{1+2\sqrt{1+\frac{1}{n}\arctan\left(\frac{k}{n}\right)}}$$

(Dan Sitaru)

86 Evaluate the limit

$$\Omega=\lim_{n\to+\infty}\sum_{k=1}^n\arcsin\frac{1}{\sqrt{n^2+k}}$$

87 Let φ denote Euler's totient function. Evaluate the limit

$$\ell=\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^n\sin\left(\frac{\pi k}{n}\right)\varphi(k)$$

88 Let $\alpha > 0$. Prove that:

$$\lim_{n\to+\infty} \frac{1}{\log n} \sum_{1\leqslant k\leqslant n^{a}} \frac{1}{k}\left(1-\frac{1}{n}\right)^{k} = \min\{1, a\}$$

89 Let us denote with ζ the Riemann zeta function with $\zeta(0) = -\frac{1}{2}$. Let us also denote with $\zeta^{(n)}$ the $n-\text{th}$ derivative of zeta. Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{\zeta^{(n)}(0)}{n!}$$

⛁

90 Let ζ denote the Riemann zeta function. Evaluate the limit

$$\ell = \lim_{n\to+\infty} n\left(\zeta(2) - \sum_{k=1}^{n}\frac{1}{k^2}\right)$$

91 Evaluate the limit

$$\ell = \lim_{n\to+\infty} \frac{(n!)^2}{(1+1^2)(1+2^2)\cdots(1+n^2)}$$

92 Evaluate the limit

$$\ell = \lim_{a\to 0} \frac{1}{a^3}\int_0^{a} \log\left(1+\tan a \tan x\right)\, dx$$

⛁

93 Let Γ denote the Euler's Gamma function. Prove that

$$\frac{\Gamma\left(\frac{1}{10}\right)}{\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{7}{15}\right)} = \frac{\sqrt{5}+1}{3^{1/10}2^{6/5}\sqrt{\pi}}$$

94 Let $f : [0, +\infty) \to \mathbb{R}$ be an integrable and uniformly continuous function. Prove that $\lim\limits_{x\to+\infty} f(x) = 0$. Does this result hold if we drop the assumption of the *uniformly continuous* ? Explain your answer.

95 Define a function $f : \mathbb{R} \to \mathbb{R}$ such that for every $q \in \mathbb{Q}$ must hold $f(q) \in \mathbb{Q}$ but $f'(q) \notin \mathbb{Q}$.

⛁The above limit tells us that $\zeta^{(n)}(0) \sim -n!$.

⛁You might as well evaluate the integral first by making the substitution $y = a - x$.

96 Let $f : [0, 1] \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} 0 & , \quad x \in [0,1] \cap (\mathbb{R}\setminus\mathbb{Q}) \\ x_n & , \quad x = q_n \in [0,1]\cap\mathbb{Q} \end{cases}$$

where x_n is a sequence such that $\lim x_n = 0$ and $0 \leqslant x_n \leqslant 1$ and q_n be an enumeration of the rationals of the interval $[0, 1]$. Prove that f is Riemann integrable and that $\int_0^1 f(x)\, dx = 0$.

97 Let f be holomorphic on the open unit disk $\mathbb{D}$ and suppose that

$$\iint\limits_{\mathbb{D}} |f(z)|^2 \, d(x, y) < +\infty$$

If the Taylor expansion of f is of the form $\sum\limits_{n=0}^{\infty} a_n z^n$ then prove that the series $\sum\limits_{n=0}^{\infty} \frac{|a_n|^2}{n+1}$ converges.

98 Let f_n be a sequence of real valued $\mathcal{C}^1$ functions on $[0, 1]$ such that forall $n \in \mathbb{N}$ the following hold:

- $|f_n'(x)| \leqslant \dfrac{1}{\sqrt{x}} \quad (0 < x \leqslant 1)$
- $\int_0^1 f_n(x)\, dx = 0$

Prove that f_n has a convergent subsequence that converges uniformly on $[0, 1]$.

99 Let $\chi_{\mathbb{Q}}$ denote the characteristic function of the rationals in $[0, 1]$. Does there exist a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ such that f_n converges to $\chi_{\mathbb{Q}}$ pointwise?

100 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\, dt = \int_0^1 t f(t)\, dt = 1 \tag{1}$$

Prove that $\int_0^1 f^2(t)\, dt \geqslant 4$.

101 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\, dt = \int_0^1 t f(t)\, dt \tag{1}$$

Prove that there exists a $c \in (0, 1)$ such that

$$\int_0^c f(t)\, dt = \frac{c}{2}\int_0^c f(t)\, dt$$

102 Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(t)\,dt = \int_0^1 tf(t)\,dt \tag{1}$$

Prove that there exists a $c \in (0,1)$ such that

$$cf(c) = 2\int_c^0 f(t)\,dt$$

103 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = f^2(x)f(-x) \tag{1}$$

Find an explicit formula for f.

104 Let $f : [0,1] \to \mathbb{R}$ be a continous function such that $\int_0^1 f(t)\,dt = 1$ and

$$\int_0^1 (1 - f(x))\, e^{-f(x)}\,dx \leqslant 0 \tag{1}$$

Prove that $f(x) = 1$ forall $x \in \mathbb{R}$.

105 Let $f : [a,b] \to [0,+\infty)$ be a continous and not everywhere 0 function. Prove that

$$\lim_{n\to+\infty} \frac{\int_a^b f^{n+1}(t)\,dt}{\int_a^b f^n(t)\,dt} = \sup_{x\in[a,b]} f(x)$$

106 Examine if there exists a continuous function $f : [1,+\infty) \to \mathbb{R}$ such that $f(x) > 0$ forall $x \in [1,+\infty)$ and $\int_1^\infty f(t)\,dt$ converges whereas $\int_1^\infty f^2(t)\,dt$ diverges.

107 Let $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and consider the function

$$f(x) = a_1 \tan x + a_2 \tan\frac{x}{2} + \cdots + a_n \tan\frac{x}{n}$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $n \in \mathbb{N}$. If $|f(x)| \leqslant |\tan x|$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then prove that

$$\left|a_1 + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right| \leqslant 1$$

Do the same exercise with the extra assumption that f is uniformly continuous.

108 Let $f : [0,1] \to \mathbb{R}$ be a twice differentiable function with a continuous second derivative. If n is a natural number greater than 1 such that

$$\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right) = -\frac{f(0)+f(1)}{2}$$

then prove that

$$\left(\int_0^1 f(t)\,dt\right)^2 \leqslant \frac{1}{5!n^4}\int_0^1 \left(f''(t)\right)^2\,dt$$

109 Prove that every function $f : \mathbb{Q} \to \mathbb{Q}$ can be written as the sum of two $1-1$ functions $g, h : \mathbb{Q} \to \mathbb{Q}$.

110 Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that any rational number is its period but any irrational is not. Also, prove that there exists no function $g : \mathbb{R} \to \mathbb{R}$ such that any irrational is its period and any rational is not.

111 Prove that the function

$$f(x) = \begin{cases} \sin\left(\ln^2 x\right) & , \quad x > 0 \\ 0 & , \quad x = 0 \end{cases}$$

has a primite on $[0,+\infty)$.

(Constanza , 2009)

112 Let $\mathbb{F}$ be an ordered field. Define $f : \mathbb{F} \to \mathbb{F}$ such that it satisfies

$$|f(x) - f(y)| \leqslant |x-y|^2, \quad \forall x, y \in \mathbb{F}$$

Is $\mathbb{F}$ necessarily Archimidean?

113 Compute the limit:

$$\ell = \lim_{n\to+\infty} \frac{1}{n^2} \sum_{1\leqslant i\leqslant j\leqslant n} \ln\left(\frac{3n-i}{3n+i}\right)\ln\left(\frac{3n-j}{3n+j}\right)$$

114 Compute the limit

$$\ell = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k+n}{n+2\sqrt{n^2+n+k}}$$

115 Compute the limit

$$\ell = \lim_{n\to+\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{i^2+j^2} - \frac{\pi\log n}{2}\right]$$

116 Let ζ denote the Riemann zeta function. Prove that

$$\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\frac{\zeta(\frac{3}{2}+it)}{\zeta(\frac{3}{2}-it)}\,dt = \frac{1}{\zeta(3)}$$

117 Let $\lfloor\cdot\rfloor$ denote the floor function. Prove that forall $n\in\mathbb{N}$ it holds that

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{k^3}\right)^{-1}\right\rfloor = 2n(n-1)$$

118 (a) Let $a>0$. Evaluate the integral

$$\mathcal{J}(a) = \int_0^a \log\left(1+\tan a\tan x\right)\,dx$$

(b) Evaluate the limit $\displaystyle\lim_{a\to 0}\frac{\mathcal{J}(a)}{a^3}$.

119 Prove that for an entire function f holding $\lim_{z\to\infty}\frac{f(z)}{z}=0$ then f is constant.

120 Let $f:\mathbb{C}\to\mathbb{C}$ be an analytic and $1-1$ function and let $\mathbb{D}$ be the open unit disk. Prove that

$$\iint_{\mathbb{D}} \left|f'(z)\right|\,dz = \operatorname{area}(f(\mathbb{D}))$$

[1]

121 Let $n\in\mathbb{N}$ and f be an entire function. Prove that for any arbitrary positive numbers a, b it holds that

$$\frac{\int_0^{2\pi} e^{-int}f\left(z+ae^{it}\right)\,dt}{\int_0^{2\pi} e^{-int}f\left(z+be^{it}\right)\,dt} = \left(\frac{a}{b}\right)^n$$

122 Let $a, b\in\mathbb{C}$ such that $|b|<1$. Prove that

$$\frac{1}{2\pi}\oint_{|z|=1}\left|\frac{z-a}{z-b}\right|^2|dz| = \frac{|a-b|^2}{1-|b|^2}+1$$

[1] This is known as Lusin Area Integral Formula.

123 Define

$$f(z) = \frac{1}{z}\cdot\frac{1-2z}{z-2}\cdots\frac{1-10z}{z-10}$$

Evaluate the contour integral $\displaystyle\oint_{|z|=100} f(z)\,dz$.

124 Prove that there does not exist a sequence $\{p_n(z)\}_{n\in\mathbb{N}}$ of complex polynomials such that $p_n(z)\to\frac{1}{z}$ uniformly on $\mathcal{C}_R=\{z\in\mathbb{C}\mid |z|=R\}$.

125 Let f be a meromorphic function on a (connected) Riemann Surface X. Show that the zeros and the poles of f are isolated points.

126 Let us prove that $0=1$. We begin by stating Picard's Little Theorem:

> **Theorem**
>
> If a function $f:\mathbb{C}\to\mathbb{C}$ is entire and non-constant, then the set of values that $f(z)$ assumes is either the whole complex plane or the plane minus a single point.

Let us now consider $g(z)=e^z$ which is definitely complex differentiable. Since the composition of complex differentiable functions is also complex differentiable then the function

$$f(z) = g\left(g(x)\right) = e^{e^z}$$

is also complex differentiable. Also, f is not constant; that is for sure. Since there exists no z such that $e^z=0$ then 0 and 1 are not in the range of f. However, this is an obscurity unless $0=1$.

Find the flaw in the above argument. [2]

127 Let $A\subseteq\mathbb{R}$ be a set of finite measure.

(a) Find the Fourier series of $|\sin\lambda x|$.

(b) Evaluate the limit

$$\ell = \lim_{\lambda\to+\infty}\int_A |\sin\lambda x|\,dx$$

[2] The flaw is not in the theorem!

128 Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^m$. Evaluate the integral

$$\mathcal{M} = \int\limits_{\mathbb{R}^m} \exp\left(-(\langle x, \mathcal{S}^{-1}x\rangle)^a\right)\, dx$$

where $\mathcal{S}$ is a positive symmetric $m \times m$ matrix and $a > 0$.

129 Let $\psi^{(n)}$ denote the n - th polygamma function and let $n \in \mathbb{N} \cup \{0\}$. Prove that

$$\frac{\psi^{(n)}(z)}{\psi^{(n+1)}(z)} \geqslant \frac{\psi^{(n+1)}(z)}{\psi^{(n+2)}(z)} \quad , \quad z > 0$$

130 Consider the points $O(0,0)$ and $A(1,0)$. Let $\Gamma(x,y)$ be a point of the plane such that $y > 0$. Set $\varphi(x,y)$ to be the angle that is defined by $O\Gamma$ and $A\Gamma$. (the one that is less than π.) Prove that the function $\varphi(x,y)$ is harmonic.

131 Let f be analytic in the unit disk $\mathbb{D}$. Suppose that $\mathrm{Re}(f(z)) \geqslant 0$ forall $z \in \mathbb{D}$ and that $f(0) = 1$. Prove that

$$\frac{1-|z|}{1+|z|} \leqslant \mathrm{Re}\left(f(z)\right) \leqslant |f(z)| \leqslant \frac{1+|z|}{1-|z|}$$

Multivariable Calculus

132 Given the curve $\gamma(t) = e^{-t}(\cos t, \sin t)$, $t \geqslant 0$

(a) Sketch its graph.

(b) Evaluate the length of the curve as well as the following line integrals

$$\text{(i)} \oint\limits_{\gamma} (x^2+y^2)\, ds \qquad \text{(ii)} \oint\limits_{\gamma} (-y, x) \cdot d(x,y)$$

(Question from a Real Analysis Exam University of Ioannina , Greece)

The $a = 1$ case can be interpreted as (the appropriate constant multiple of) the density of a multivariate normal distribution.

Actually the above inequality is a consequence of a stronger one namely this:

$$\psi^{(m)}(z)\psi^{(n)}(z) \geqslant \psi^{\left(\frac{m+n}{2}\right)}(z)$$

whenever $\frac{m+n}{2} \in \mathbb{N}$. The proof of it may be found at Joy of Mathematics.

133 (a) Let $\mathbb{D} \subset \mathbb{R}^2$ be the unit disk and $\partial\mathbb{D}$ be its positive oriented boundary. Evaluate the following line integral

$$\oint\limits_{\partial\mathbb{D}} \left(x - y^3, x^3 - y^2\right) \cdot d(x,y)$$

(b) Can you deduce if the function

$$\bar{f}(x,y) = \left(x - y^3, x^3 - y^2\right)$$

is a vector field by basing your reasoning **solely** on question **(a)** ?

(Question from a Real Analysis Exam University of Ioannina , Greece)

134 (a) Let $f \in \mathcal{C}^2(\mathbb{R})$ such that $\operatorname{div} \operatorname{grad}(f) = 0$ and $\mathbb{D} \subseteq \mathbb{R}^2$ be a $\mathcal{C}^1$ normal set. Prove that

$$\oint\limits_{\partial\mathbb{D}} \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right) \cdot d(x,y) = 0$$

(b) Examine if

$$\bar{f}(x,y) = (2x\cos y, -x^2 \sin y)$$

is a conservative field and if so, find a scalar potential.

(Question from a Real Analysis Exam University of Ioannina , Greece)

135 Prove that for every $c > 0$ the set

$$\mathcal{B}_{f,g} = \{(x,y,z) \in \mathbb{R}^3 : (x - f(z))^2 + (y - g(z))^2 \leqslant c \ , \ z \in [a,b]\}$$

has the same volume for every function $f, g : [a,b] \to \mathbb{R}$.

136 Consider the subset of $\mathbb{R}^3$

$$\mathcal{B} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant a\}, \ a > 0$$

(a) Evaluate

(i) the volume of $\mathcal{B}$.

(ii) the triple integral

$$\mathcal{T} = \iiint_{B} (x^2+y^2)z\,\mathrm{d}(x,y,z)$$

(iii) the area of the boundary of $\mathcal{B}$.

(iv) the surface integral

$$\mathcal{S} = \oint_{\partial B} \sqrt{1+4z^2}\,\mathrm{d}\sigma$$

(b) Express the volume of $\mathcal{B}$ through a suitable continuously differentiable $G : \mathbb{R}^3 \to \mathbb{R}^3$ and through a suitable surface integral.

137 Prove that the work

$$\mathcal{W} = -\oint_{\gamma} \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \cdot \mathrm{d}(x,y,z)$$

produced along a $\mathcal{C}^1$ oriented curve γ of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ depends only on the distances of starting and ending point of γ about the origin.

138 Let $\mathcal{V}_n(R)$ be the volume of the ball of center 0 and radius $R>0$ in $\mathbb{R}^n$. Prove that for $n \geqslant 3$ it holds that

$$\mathcal{V}_n(1) = \frac{2\pi}{n}\mathcal{V}_{n-2}(1)$$

139 Let $\mathbb{S}$ denote the area bounded by the curves $x^2y=1$ and $x^2y=2$ as well as the lines $y=x$ and $y=2x$ and let γ denote its negative oriented boundary. Evaluate

$$\mathcal{J} = \oint_{\gamma} \left(e^{-x^2}-6y\right)\,\mathrm{d}x + \left(4x-7y^7\right)\,\mathrm{d}y$$

140 Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a continously differentiable function and let $\mathcal{C}_r$ be the circle of origin $(0,0)$ and radius $r>0$. Prove that:

$$\frac{1}{2\pi}\lim_{r\to 0}\frac{1}{r}\oint_{\mathcal{C}_r} u\,\mathrm{d}s = u(0,0)$$

141 Let $f(\mathbf{x}) = \mathbf{x}^\top Q\mathbf{x}$ where $\mathbf{x}^\top = (x_1,\dots,x_n) \in \mathbb{R}^n$ and Q is the diagonal matrix

$$Q = \begin{pmatrix} q_1 & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q_n \end{pmatrix} \qquad q_i \in \mathbb{R},\ i=1,\dots,n$$

(a) Give the derivative as well as the Hessian matrix of f.

(b) Give conditions for the q_i such that f has **a)** a local maximum **b)** a local minumum and **c)** neither of the previous ones.

(c) Compute the Taylor polynomial of degree k of f around $\mathbf{x}=\mathbf{0}$ forall $k \in \mathbb{N}$.

142 Let $\mathcal{S} = [0,1]\times[0,1] \subset \mathbb{R}^2$. Evaluate the integral

$$\mathcal{J} = \iint_{\mathcal{S}} \max\{x,y\}\,\mathrm{d}(x,y)$$

Hint: It holds that

$$\max\{x,y\} = \begin{cases} x &, \quad 0 \leqslant y \leqslant x \leqslant 1 \\ y &, \quad 0 \leqslant x \leqslant y \leqslant 1 \end{cases}$$

Hence

$$\begin{aligned} \int_0^1\int_0^1 \max\{x,y\}\,\mathrm{d}(x,y) &= \int_0^1\int_0^x x\,\mathrm{d}(y,x) + \\ &\quad + \int_0^1\int_0^y y\,\mathrm{d}(x,y) \\ &= 2\int_0^1\int_0^x x\,\mathrm{d}(y,x) \\ &= 2\int_0^1 x^2\,\mathrm{d}x \\ &= \frac{2}{3} \end{aligned}$$

143 Let M be the interesection of the elliptic cylinder $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} \leqslant 1$ and the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1 \quad a>0,\ b>0,\ c>0$$

For all $n \in \mathbb{N}$ evaluate the integrals

$$I_n = \iiint_{M} \left(a^2b^2-b^2x^2-a^2y^2\right)^{n-\frac{1}{2}}\,\mathrm{d}(x,y,z)$$

An interpretation of this integral; if you have two independent uniform $(0,1)$ random variables, the expected value of the maximum is $\frac{2}{3}$. (And the expected value of the minimum is $\frac{1}{3}$.) More generally: if you have n independent uniform $(0,1)$ random variables, the expected value of the maximum is $\frac{n}{n+1}$. In more detail: if you order these random variables after the fact so that $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_n$, then the expected value of Y_k is $\frac{k}{n+1}$. (The general name for this sort of reasoning is order statistics.)

(Question from a Real Analysis Exam University of Ioannina, Greece)

144 Let $\mathcal{C} = [0,1] \times [0,1] \times \cdots \times [0,1] \subseteq \mathbb{R}^n$ be the unit cube. Define the function

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1 x_2 \cdots x_n}{x_1^{a_1} + x_2^{a_2} + \cdots + x_n^{a_n}}$$

where a_i arbitrary positive constants. For which values of $a_i > 0$ is the value of the integral $\int_{\mathcal{C}} f$ finite?

145 Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) \geqslant 0$ and $\int_{-\infty}^{\infty} f(t)\,dt = 1$. For $r \geqslant 0$ we define

$$I_n(r) = \int \cdots \int_{x_1^2 + x_2^2 + \cdots + x_n^2 \leqslant r} f(x_1) f(x_2) \cdots f(x_n)\, d(x_1, x_2, \ldots,$$

Evaluate $\lim I_n(r)$.

146 Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\mathbb{A}$ denote the area measure on $\mathbb{D}$ normalised so that $\mathbb{A}(\mathbb{D}) = \pi$. Verify or disprove that

$$\iint_{\mathbb{D}} \left| \log\left(\frac{e}{1-z}\right) \right|^2 dA = \frac{\pi^3}{6}$$

147 For a given function $f : \mathbb{R}^3 \to \mathbb{R}^3$ the $\int_{\mathbb{R}^3} |f(x)|\,dx$ exists. If for every plane $\mathcal{P}$ of $\mathbb{R}^3$ it holds that $\int_{\mathcal{P}} f(x)\,ds = 0$ then prove that f is the zero function.

148 Let $A \subseteq \mathbb{R}^n$. If A is Jordan measurable and has zero measure prove that

$$\int_A 1\, d\bar{x} = 0$$

General Topology

149 Find a countable and dense subset of $\mathbb{R} \setminus \mathbb{Q}$ with respect to the usual topology.

As a hint you may use Fourier transform.

150 Let $\mathcal{X} = [0, +\infty) \cup \{+\infty\}$. We endow it with the metric

$$\rho(x, y) = |\arctan x - \arctan y|$$

Prove that under this metric $\mathcal{X}$ is separable , complete and compact.

151 Does there exist an enumeration $\{q_n \in \mathbb{Q} : n \in \mathbb{N}\}$ of $\mathbb{Q}$ such that

$$\mathbb{R} \neq \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n}, q_n + \frac{1}{n}\right)$$

152 Prove that there does not exist an $1-1$ and continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}$.

153 Let Ω be a metric space. Suppose that every bounded subset of Ω has at least one accumulation point. Prove that Ω is a complete metric space.

154
(a) Let (X, ρ) be a compact metric space and let $f : X \to X$ be an isometry. Prove that f is onto.
(b) Prove that the ℓ^2 space (that is the space of the sequences for which $\sum_{n=1}^{\infty} x_n^2$ converges) is not compact endowed by the metric

$$\rho(x_n, y_n) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$$

155 Prove that there exists no continuous and $1-1$ map (depiction) from a sphere to a proper subset of it.

156 Is the set $\mathcal{S} = \mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$ complete? Give a brief explanation.

157 Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Endow it with the metric

$$d(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

(a) Show that the sequence $a_n = n$ is a Cauchy one.
(b) Is the sequence $\frac{1}{n}$ a Cauchy one?
(c) Show that any sequence a_n in $\mathbb{R}^+$ converges in $\mathbb{R}^+$ in the metric d above if and only if it converges in $\mathbb{R}$ in the standard metric $|x - y|$ and that the limits in the two cases are equal.

158 Let us define the following function:

$$f(x) = \begin{cases} x & , \quad 0 \leqslant x < 1 \\ 1 & , \quad x > 1 \end{cases}$$

as well as $d_m(x, y) = f(|x - y|)$.

(a) Show that d_m is a metric on $\mathbb{R}$. *You may call it the mole metric. If points are close (closer than one meter), their distance is the usual one, but are they far apart (more than one meter) we do not distinguish between their distances; they are just far apart.*

(b) Show that $\mathbb{R}$ endowed with the above metric is complete and bounded but not compact. Is it totally bounded? Why / Why not?

159 Prove that the set $\mathbb{R}^2 \setminus \{0, 0\}$ is not simply connected.

160 Find a sequence of open sets $\{G_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}$ such that

$$\mathbb{Z} = \bigcap_{n=1}^{\infty} G_n$$

161 (a) Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Prove that the set

$$\mathcal{D}(\theta) = \left\{ (\cos 2n\pi\theta, \sin 2n\pi\theta) \in \mathbb{R}^2 : n \in \mathbb{N} \right\}$$

is a dense subset of the circle $\mathbb{S}^1 : x^2 + y^2 = 1$.

(b) Find a countable and dense subset of $\mathbb{R} \setminus \mathbb{Q}$ with respect to the usual metric.

162 Let us denote $\mathbb{S}^2$ the unit sphere that is the set

$$\mathbb{S}^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \middle| x^2 + y^2 + z^2 = 1 \right\}$$

If $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a continuous function such that $f(x) \neq f(-x)$ forall $x \in \mathbb{S}^2$ then prove that f is onto.

Well , the problem actually is not of an analysis nature but that of Algebraic Topology. Try to construct a deformation retraction from $\mathbb{R}^2 \setminus \{0, 0\}$ to $\mathbb{S}^1$ (the unit circle). For example take $f(x) = \dfrac{x}{\|x\|}$. Then the fundamental groups are isomorphic , however $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ and hence the fundamental group is not trivial. Therefore , the set is not simply connected.

Simply take

$$G_n = \bigcup_{m \in \mathbb{Z}} \left(m - \frac{1}{n}, m + \frac{1}{n} \right)$$

163 Examine if there exist non constant functions $f : \mathbb{R} \to \mathbb{R}$ that map any open interval onto a closed one.

164 Let (X, d) be a complete and a compact metric space. Prove that there exists a unique number $r = r(X, d)$ with the property:

For all $n \in \mathbb{N}$ and for all $x_i, i = 1, 2, \ldots, n$ there exists $z \in X$ such that $\dfrac{1}{n} \sum\limits_{i=1}^{n} d(z, x_i) = r$.

165 Prove that a metric space (X, d) containing infinite points, where d is the discrete metric, is not compact.

166 Prove that the set

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid y \cos x + x \sin y = 1 \right\}$$

is not path-connected with respect to the relative topology of $\mathbb{R}^2$.

Integrals and Series

167 Evaluate

$$\mathcal{J} = \int_1^{\infty} \sum_{n=0}^{\infty} \frac{-dx}{(n + x)^3}$$

168 Let $a \geqslant -1$. Evaluate

$$\mathcal{J} = \int_0^{\pi/2} \log\left(1 + a \sin^2 x\right) dx$$

169 Let $n \in \mathbb{N} \mid n > 2$. Prove that

$$\int_0^{\infty} \frac{\log\left(\frac{1}{x}\right)}{(1 + x)^n} dx = \frac{1}{n - 1} \sum_{k=1}^{n-2} \frac{1}{k}$$

170 Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\arctan \dfrac{x}{x + 1}}{\arctan \dfrac{1 + 2x - 2x^2}{2}} dx$$

(Russian Mathematical Olympiad)

171 For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

(Putnam 2001)

172 Prove that

$$\int_0^1 \prod_{n=1}^{\infty} (1 - x^n) \, dx = \frac{4\pi\sqrt{3} \sinh \frac{\pi\sqrt{23}}{3}}{\sqrt{23} \cosh \frac{\pi\sqrt{23}}{2}}$$

173 Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\arctan \sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}} \, dx$$

174 Let $a \in \mathbb{R}$. Evaluate the integral

$$\mathcal{J} = \int_{-\infty}^{\infty} \frac{\cos ax}{e^x + e^{-x}} \, dx$$

175 Evaluate the integral

$$\mathcal{J} = \int_0^{\infty} \frac{x^2 - 4}{x^2 + 4} \frac{\sin 2x}{x} \, dx$$

176 Evaluate the double series

$$\mathcal{S} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}$$

(Putnam 2016)

This integral is known with the name "Ahmed's integral".

The evaluation of this integral allows to tell that

$$\mathfrak{Re}\left[\psi^{(0)}\left(\frac{3}{4} - \frac{ia}{4}\right) - \psi^{(0)}\left(\frac{1}{4} - \frac{ia}{4}\right)\right] = \pi \operatorname{sech}\left(\frac{\pi a}{2}\right)$$

where $\psi^{(0)}$ is the digamma function.

177 Evaluate the integral

$$\mathcal{J} = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right) dx$$

178 Let $\psi^{(1)}$ denote the trigamma function. Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} (-1)^{n-1} \left(\psi^{(1)}(n)\right)^2$$

(Cornel Ioan Valean)

179 Let Li_2 denote the dilogarithm function and Γ denote the Gamma function. Prove that

$$\int_0^1 \left(\mathrm{Li}_2\left(e^{-2\pi i x}\right) + \mathrm{Li}_2\left(e^{2\pi i x}\right)\right) \log \Gamma(x) \, dx = \frac{\zeta(3)}{2}$$

where ζ is the Riemann zeta function.

180 Let Li_2 denote the dilogarithm function. Prove that

$$\int_0^{\infty} \mathrm{Li}_2\left(e^{-\pi x}\right) \arctan x \, dx = \frac{\pi^2}{18} - \frac{3\zeta(3))}{8}$$

181 Prove that

$$\sum_{n=1}^{\infty} \arctan\left(\frac{10n}{(3n^2+2)(9n^2-1)}\right) = \ln 3 - \frac{\pi}{4}$$

182 Let ζ denote the Riemann zeta function. Prove that

$$\sum_{k=1}^{\infty} \frac{k\zeta(2k)}{4^{k-1}} = \frac{\pi^2}{4}$$

183 Let Li_3 denote the trilogarithm function. Prove that

$$\sum_{n=1}^{\infty} \mathrm{Li}_3\left(e^{-2n\pi}\right) = \frac{7\pi^3}{360} - \frac{\zeta(3)}{2}$$

One can also evaluate the general form

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right)^m dx \quad m \geqslant 1$$

(Seraphim Tsipelis)

184 Prove that

$$\int_0^{2-\sqrt{3}} \frac{\arctan t}{t}\, dt = \frac{\pi}{12}\log\left(2-\sqrt{3}\right) + \frac{2\mathcal{G}}{3}$$

where $\mathcal{G}$ denotes the Catalan constant.

185 Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+1)} = 1-\gamma$$

where γ stands for the Euler - Mascheroni constant.

(Seraphim Tsipelis, Kotronis Anastasios)

186 Evaluate the following double series

$$\mathcal{S} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (-1)^{m+n} \frac{m\ln(m+n)}{(m+n)^3}$$

(Enkel Hysnelaj)

187 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n}\left(\zeta(2) - \sum_{k=1}^{n}\frac{1}{k^2}\right) = \frac{7\zeta(4)}{4}$$

where ζ is the Riemann zeta function.

188 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{n}\cos\left(\frac{n\pi}{3}\right) = -\frac{\pi^2}{36}$$

189 Prove that

$$\sum_{j=2}^{\infty}\prod_{k=1}^{j}\frac{2k}{j+k-1} = \pi$$

190 *This series may be called as "The harmony of the harmony"*. Evaluate the series

$$\sum_{n=1}^{\infty}\frac{1}{(n+2)2^{n+2}}\sum_{k=1}^{n}\frac{1}{k+1}\sum_{m=1}^{k}\frac{1}{m} = \frac{\ln^3 2}{6}$$

191 Let $\mathbb{Z} \ni k \geqslant 1$. Prove that

$$\int_0^1 \ln^k(1-x)\ln x\, dx = (-1)^{k+1}k!\left(k+1-\sum_{m=2}^{k+1}\zeta(k)\right)$$

where ζ denotes the Riemann zeta function.

(Ovidiu Furdui)

192 Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3}}{9-4n^2}$$

193 Let $r \in \mathbb{R}$. Prove that

$$\sum_{n=-\infty}^{\infty}\arctan\left(\frac{\sinh r}{\cosh n}\right) = \pi r$$

(H. Ohtsuka)

194 Evaluate

$$\int_{-\infty}^{\infty}\frac{\arctan x}{x^2+x+1}\, dx$$

195 Let Γ denote the Gamma function. Evaluate the integral

$$\int_0^1 \Big(\log\Gamma(x) + \log\Gamma(1-x)\Big)\log\Gamma(x)\, dx$$

196 Evaluate the integrals

$$\text{(i)}\ \int_0^{\infty}\frac{\ln x}{e^x+1}\, dx \qquad\qquad \text{(ii)}\ \int_0^{\infty}\frac{\ln x}{e^x-1}\, dx$$

197 Let erf denote the error function. Prove that

$$\int_0^{\infty} e^{-x}\operatorname{erf}^2(x)\, dx = \frac{2\sqrt{2}}{\pi}\arctan\frac{1}{\sqrt{2}}$$

The more general identity

$$\prod_{n=-\infty}^{\infty}\left(1+\frac{\sin r}{\cosh n}\right) = e^{\pi r - r^2}$$

for $\mathfrak{Re}(r)=0$ seems to be true as pointed out by Tintarn at AoPS.com.

198 Evaluate

$$\int_0^\infty \left(\frac{x}{e^x - e^{-x}} - \frac{1}{2} \right) \frac{dx}{x^2}$$

199 Prove that

$$\int_0^1 \frac{\log(1+x)\log^2 x}{1-x}\, dx = \frac{7}{2}\log 2\zeta(3) - \frac{19}{720}\pi^4$$

(Cornel Ioan Valean)

200 Let $\mathcal{H}_n$ denote the n-th harmonic number. Evaluate the sum

$$\mathcal{S} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{k+n} \frac{\mathcal{H}_{k+n}^2}{k+n}$$

(Cornel Ioan Valean)

201 Calculate

$$\mathcal{S} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1 + k\log k}{2 + (k+1)\log(k+1)}$$

202 Calculate

$$\mathcal{S} = \sum_{n=1}^{\infty} \arctan(\sinh n) \arctan\left(\frac{\sinh 1}{\cosh n} \right)$$

(H. Ohtsuka)

203 Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\mathcal{J} = \int_0^{\pi/2} \frac{\{\tan x\}}{\tan x}\, dx$$

204 Calculate

$$\mathcal{J} = \int_0^{\pi/2} x \ln \tan x \, dx$$

205 Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\, dx = \frac{\gamma}{2}$$

206 Calculate

$$\int_0^\infty \frac{\log x}{(2x+1)\left(x^2+x+1\right)}\, dx$$

207 Let $\{\cdot\}$ denote the fractional part. Evaluate

$$\int_0^1 \left\{ \frac{1}{x} \right\}^2 \left\{ \frac{1}{1-x} \right\} dx$$

208 Let Ω denote the root of the equation $xe^x = 1$. Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{\left(e^x - x\right)^2 + \pi^2} = \frac{1}{1+\Omega}$$

209 Evaluate the series

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} \frac{x^2}{n^2+n-1}$$

as well as the product

$$\Pi = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2+n-1} \right)$$

210 Let ζ denote the Riemann zeta function. Prove the identity:

$$\frac{1}{2\pi} \mathrm{Li}_2\left(e^{-2\pi}\right) = \log(2\pi) - 1 - \frac{5\pi}{12} - \sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{n(2n+1)}$$

where Li_2 denotes the dilogarithm function.

211 Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)4^n} = \mathcal{G}$$

where ζ denotes the Riemann zeta function and $\zeta(0) = -\frac{1}{2}$.

212 Let $s \in \mathbb{C}$ such that $\mathfrak{Re}(s) > 1$. Evaluate the following double Euler sum

$$\mathcal{S} = \sum_{(j,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{\left(j^2+k^2\right)^s}$$

213 Evaluate the integral

$$\mathcal{J} = \int_0^{\pi/2} \sin^2 x \log\left(\sin^2(\tan x)\right)\, dx$$

214 Let $0 \leqslant \alpha, \beta \leqslant \pi$ and $\kappa > 0$. Prove that

$$\int_0^\infty \frac{1}{x} \log\left(\frac{x^2 + 2\kappa x \cos\beta + \kappa^2}{x^2 + 2\kappa x \cos\alpha + \kappa^2}\right) dx = \alpha^2 - \beta^2$$

215 Let γ denote the Euler – Mascheroni constant. Define $F(x) = \sum\limits_{n=1}^{\infty} x^{2^n}$. Prove that

$$\gamma = 1 - \int_0^1 \frac{F(x)}{1+x}\, dx$$

216 Let $\mathcal{B}_n$ denote the n -th Bernoulli number. Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mathcal{B}_{2n} x^{2n}}{2n\,(2n)!} = \log\frac{x}{2} - \log\sin\frac{x}{2}$$

217 Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{1-x}{\log x} \sum_{n=0}^{\infty} x^{2^n}\, dx$$

218 Prove that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56700}$$

219 Evaluate the sum

$$\mathcal{S} = \sum_{n=-\infty}^{\infty} \frac{\log\left|n + \frac{1}{4}\right|}{n + \frac{1}{4}}$$

(Seraphim Tsipelis)

220 Let $\mathcal{H}_n$ denote the n-th harmonic sum. Evaluate the sum:

$$\mathcal{S} = \sum_{n=1}^{\infty} \left(\mathcal{H}_n - \log n - \gamma - \frac{1}{2n} + \frac{1}{12n^2}\right)$$

(M. Omarjee)

221 Prove that

$$\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1}\binom{n}{k}}\right)^{\frac{n(n+1)}{2^{n+3}}} = e^{7\zeta(3)/24\zeta(2)}$$

where ζ denotes the Riemann zeta function.

222 Let $\mathbb{R} \ni s > 2$. Evaluate the (double) sum:

$$\mathcal{S} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{m^2 + 4mn + n^2}{(m^2 + mn + n^2)^s}$$

(Kent Merryfield)

223 Let $a \in [-\pi, \pi]$ and let us denote with Ci the Cosine integral function. Evaluate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\mathrm{Ci}(na)}{n^2}$$

224 Let $\alpha\,,\ \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Prove that

$$\int_0^\infty \frac{\log x}{(x+\alpha)(x+\beta)}\, dx = \frac{1}{2(\beta-\alpha)}\left[\log^2\beta - \log^2\alpha\right]$$

(Grigorios Kostakos)

225 Let γ denote the Euler - Mascheroni constant. Prove that

$$\int_0^\infty \frac{\cos x^n - \cos x^{2n}}{x} \log x\, dx = \frac{12\gamma^2 - \pi^2}{2(4n)^2}$$

The most straight forward approach is to use Fourier series beginning by equation (2) at the link. The final answer is

$$\mathcal{S} = \frac{\gamma\pi^2}{12} + \frac{\pi^2 \ln a}{12} - \frac{\pi^2 \ln 2}{12} - \frac{\zeta'(2)}{2} - \frac{a^2}{8}$$

where γ denotes the Euler - Mascheroni constant.

The interested reader might as well give a try the following integral

$$\mathcal{J} = \int_0^\infty \frac{\log^2 x}{(x+\alpha)(x+\beta)}\, dx$$

226 Calculate

$$\mathcal{M}=\int_0^\infty\int_0^\infty\cdots\int_0^\infty \frac{\prod\limits_{m=1}^{n}\cos(x_m)}{\sum\limits_{m=1}^{n}x_m}\,d(x_1,x_2,\ldots,x_n)$$

227 Let $\mathcal{H}_n$ denote the n -th harmonic number. Prove that $|z|<1$ it holds that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\mathcal{H}_{2k}}{2k+1}z^{2k+1}=\frac{\arctan z}{2}\log\left(1+z^2\right)$$

228 Let $\mathcal{B}_n$ denote the n-th Bernoulli number. Prove that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\mathcal{B}_{2n}x^{2n}}{2n\,(2n)!}=\log\frac{x}{2}-\log\sin\frac{x}{2}$$

229 Let $\mathcal{G}$ denote the Catalan's constant and $\mathcal{H}_n$ the n - th harmonic number. Prove that

$$\sum_{n=1}^{\infty}\left(\frac{\mathcal{H}_{4n-3}}{4n-3}-\frac{\mathcal{H}_{4n-2}}{4n-2}\right)=\frac{\pi^2}{64}+\frac{\pi\log 2}{32}+\frac{\mathcal{G}}{2}-$$

$$-\frac{3\log^2 2}{16}-\frac{3\pi\log 2}{32}$$

(Cornel Ioan Valean)

230 Let $\mathbb{A}$ denote the Glashier - Kinkelin constant and γ the Euler - Mascheroni constant. Prove that

$$\prod_{k=1}^{\infty}\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}(k+n+m)^{\frac{(-1)^{k+m+n}}{k+m+n}}=$$

$$=\frac{\mathbb{A}^{3/2}}{\pi^{3/4}e^{1/8-(7/12+\gamma)\log 2+\frac{1}{2}\log^2 2}}$$

(Cornel Ioan Valean)

Currently I do not have a solution on this but the most straight forward idea is to actually try to find the number of ways n can be written as a sum of three numbers and reduce the triple product into a single one.

231 Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2=$$

$$=\frac{\pi^2\ln 2}{6}-\frac{\ln^3 2}{3}-\frac{3}{4}\zeta(3)$$

(Ovidiu Furdui)

232 Let k be a positive integer. Evaluate the multiple sum

$$\mathcal{S}=\sum_{i_1,\ldots,i_k\geqslant 1}\frac{1}{i_1\cdots i_k(i_1+\cdots+i_k)^2}$$

(Ovidiu Furdui)

233 Evaluate

$$\int_0^\infty\int_0^\infty\frac{d(x,y)}{(e^x+e^y)^2}$$

(Ovidiu Furdui)

234 Evaluate the integral

$$\int_0^\infty\frac{e^x-1}{e^x+1}\ln^k\left(\frac{e^x+1}{e^x-1}\right)dx$$

235 Let μ denote the Möbius function. Evaluate the series

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{(-1)^{\mu(n)}}{n^s}$$

where $\mathfrak{Re}(s)>1$.

236 Let $n\in\mathbb{N}$ and ζ denote the Riemann zeta function. Prove that

$$\int_0^{\pi/2}(\log\sin x)^n\,\tan x\,dx=(-1)^n\frac{n!\,\zeta(n+1)}{2^{n+1}}$$

For $k=1$ the sum equals $\frac{(k+1)!\zeta(k+2)}{2}$ whereas for $k\geqslant 2$ the sum equals

$$k!\left(\frac{k+1}{2}\zeta(k+2)-\frac{1}{2}\sum_{i=1}^{k-1}\zeta(k+1-i)\zeta(i+1)\right)$$

237 Let $\mathcal{G}$ denote the Catalan's constant. Prove that

$$27\sum_{n=0}^{\infty}\frac{16^n}{(2n+3)^3(2n+1)^2\binom{2n}{n}^2} = \\ = \frac{27}{2}\left(7\zeta(3)+(3-2\mathcal{G})\pi-12\right)$$

238 Let $\mathcal{H}_n$ denote the n - th harmonic number. Prove that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\mathcal{H}_n\mathcal{H}_{n+1}}{(n+1)^2}=\frac{\pi^4}{480}$$

239 Express in terms of dilogarithm the series

$$\mathcal{S}=\sum_{n=1}^{\infty}(n\operatorname{arccot} n-1)$$

240 Let lcm denote the least common multiple. Prove that for all $s>1$ it holds that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\operatorname{lcm}^s(m,n)}=\frac{\zeta^3(s)}{\zeta(2s)}$$

where ζ is the Riemann zeta function.

241 The n -th Fibonacci number is defined as $F_0=0$, $F_1=1$ and recursively via the relation

$$F_{n+2}=F_{n+1}+F_n \quad \text{for all} \quad n\geqslant 0$$

Prove that

$$\sum_{n=0}^{\infty}\arctan\left(\frac{(-1)^n}{F_{n+1}(F_n+F_{n+2})}\right)=\arctan\left(\sqrt{5}-2\right)$$

242 Let ζ denote the Riemann zeta function and let $\mathbb{N}\ni s\geqslant 2$. Prove that

$$\int_0^1 \operatorname{arctanh}^s(x)\,dx=\frac{2\zeta(s)\left(2^s-2\right)\Gamma(s+1)}{4^s}$$

The above series was proved by Jacopo D' Aurizio , an MSE user. The series goes deeper and is actually a closed form of the hypergeometric function

$${}_4F_3\left(1,1,1,\frac{3}{2};\frac{5}{2},\frac{5}{2},\frac{5}{2};1\right)$$

243 Evaluate the product

$$\Pi=\prod_{n=1}^{\infty}\left(1+\frac{1}{4n}\right)^2\left(\frac{2n+1}{2n+1+(-1)^{n-1}}\right)^{(-1)^{n-1}}$$

244 Let T_n denote the n - th triangular number. Evaluate

$$\sum_{n=1}^{\infty}\frac{1}{(8T_n-3)(8T_{n+1}-3)}$$

245 Let $\psi^{(0)}$ denote the digamma function and μ the Möbius function. Prove that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi^{(0)}\left(1+\frac{1}{n}\right)=\frac{1}{2}$$

246 Let μ denote the Möbius function. Prove that

$$\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n}=-1$$

247 Let gd denote the Gudermannian function. Evaluate the integral:

$$\mathcal{J}=\iint\limits_{[0,1]^2}\frac{\operatorname{gd}(\log xy)}{1-xy}\,d(x,y)$$

248 Let F_n denote the n -th Fibonacci number and let $\mathcal{H}_n^{(2)}$ denote the n -th harmonic number of weight 2. Evaluate the series

$$\mathcal{S}=\sum_{n=1}^{\infty}\frac{F_n\mathcal{H}_{n-1}^{(2)}}{n^2\binom{2n}{n}}$$

249 Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty}\psi^{(1)}(n)x^n=\frac{x}{1-x}\left(\zeta(2)-\operatorname{Li}_2(x)\right)$$

In continuity , investigate for which $x\in\mathbb{R}$ does the series converge .

250 Let $\psi^{(1)}$ denote the trigamma function. Prove that

$$\sum_{n=1}^{\infty}\frac{\psi^{(1)}(n)\psi^{(1)}(n+1)}{n^2}=\frac{\pi^6}{840}=\frac{9\zeta(6)}{8}$$

(Seraphim Tsipelis)

251 Let Li_2 denote the dilogarithm function. Evaluate the double integral

$$\mathcal{J} = \int_0^1 \int_0^1 \frac{\log x \log y}{(1-x)(1-y)} \frac{\mathrm{Li}_2(xy)}{xy} \, \mathrm{d}(x, y)$$

252 Evaluate the series

$$\Omega = \sum_{n=1}^{\infty} \arctan\left(\frac{9}{9+(3n+5)(3n+8)}\right)$$

(Dan Sitaru)

253 Let γ denote the Euler - Mascheroni constant and $\{\cdot\}$ the fractional function. Prove that

$$\int_0^1 \{x\} \cdot \left\{\frac{1}{1-x}\right\} \mathrm{d}x = \frac{\pi^2}{12} - \gamma$$

254 Let $\{\cdot\}$ denote the fractional function. Prove that

$$\int_1^{\infty} \frac{\{x\}}{x^5} \, \mathrm{d}x = \frac{1}{3} - \frac{\pi^4}{360}$$

255 Let $a \in \mathbb{R}$.Evaluate the integral

$$\mathcal{J} = \int_0^{\infty} \frac{\sin^2 ax}{x(1-e^x)} \, \mathrm{d}x$$

256 Let ζ denote the zeta Riemann function and Li_2 denote the dilogarithm function. Evaluate the integral

$$\int_0^1 \left[\log x \log(1-x) + \mathrm{Li}_2(x)\right] \left(\frac{\mathrm{Li}_2(x)}{x(1-x)} - \frac{\zeta(2)}{1-x}\right) \mathrm{d}x$$

257 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\sum_{n=1}^{\infty} \frac{\mathcal{H}_n^2}{n(n+1)} = 3\zeta(3)$$

The result is $4\zeta(2)\zeta(3) - 9\zeta(5)$.

258 Let ζ denote the Riemann zeta function. Define

$$\zeta^*(n) = \begin{cases} \zeta(n) & , \quad n > 1 \\ \gamma & , \quad n = 1 \end{cases}$$

where γ is the Euler - Mascheroni constant. Evalate the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(\zeta^*(n) - 1)\cos\left(\frac{n\pi}{3}\right)}{n}$$

259 Let $n, m \in \mathbb{N}$. Define:

$$\mathcal{S}_n^{(m)} = \sum_{k=0}^{n} k^m \binom{n}{k}^{-1}$$

(a) Prove that $\mathcal{S}_n^{(1)} = \frac{n}{2}\mathcal{S}_n^{(0)}$.

(b) Use (a) to deduce that

$$\mathcal{S}_{n+1}^{(0)} = \frac{n+2}{2(n+1)}\mathcal{S}_n^{(0)} + 1$$

(c) Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$

260 Let Li_4 denote the polylogarithm of order 4. Evaluate the integral

$$\mathcal{J} = \int_0^1 \frac{\log x \log(1-x) \mathrm{Li}_4(x)}{1-x} \, \mathrm{d}x$$

261 Evaluate

$$\mathcal{J} = \int_0^{\infty} \ln^2\left(\frac{x}{x^2+1}\right) \frac{1}{(x^2+1)^2} \, \mathrm{d}x$$

262 Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{4^n}{\binom{2n}{n}(4n^2-1)}$$

In fact something more general holds

$$\sum_{k=0}^{n} a^n b^{n-k} \binom{n}{k}^{-1} = \frac{n+1}{(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)^{n+1}} \sum_{k=1}^{n+1} \frac{(a^k+b^k)\left(\frac{1}{a}+\frac{1}{b}\right)^k}{k}$$

and is a consequence of a theorem named by Mansour who proved it.

263 Let L_n denote the n -th Lucas number, defined by
$L_0 = 2$, $L_1 = 1$ and for all $n \geqslant 2$

$$L_n = L_{n-1} + L_{n-2}$$

Compute the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \arctan \left(\frac{L_{n+1}^2}{1 + L_n L_{n+1}^2 L_{n+2}} \right)$$

264 Compute the multiple integral

$$\int \cdots \int_{[0,1]^n} \frac{\sum_{k=1}^{n} \log\left(1 - x_k\right) \prod_{k=1}^{n} \log(1 - x_k)}{\left(\sum_{k=1}^{n} x_k\right) \prod_{k=1}^{n} x_k} \, \mathrm{d}\left(x_1, \ldots, x_n\right)$$

265 Let $\mathcal{H}_n$ denote the n-th harmonic number. Prove that

$$\mathcal{S} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mathcal{H}_n}{kn(k+n)^3} = \frac{215}{48} \zeta(6) - 3\zeta^2(3)$$

266 Evaluate the integral

$$\mathcal{J} = \int_0^{\infty} \frac{\sin^2 ax}{x(1 - e^x)} \, \mathrm{d}x$$

267 Prove that

$$\int_{-\infty}^{\infty} \sin\left(x^2 + \frac{1}{x^2}\right) \mathrm{d}x = \sqrt{\frac{\pi}{2}} \left(\sin 2 + \cos 2\right)$$

268 Let $\gcd(\cdot, \cdot)$ denote the greatest common divisor.
Evaluate the sum

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\gcd(n, 2016)}{n^2}$$

269 Prove that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \cos \frac{k\pi}{n} \right) = -\frac{4}{5}$$

270 Prove that

$$\int_0^1 \int_0^1 \int_0^1 \frac{\mathrm{d}(x, y, z)}{\ln x + \ln y + \ln z} = -\frac{1}{2}$$

Open Problems

In this section we shall present some open problems.

1. Can we cover a unit square with $\frac{1}{k}\cdot\frac{1}{k+1}$ rectangles? Here $k \in \mathbb{N}$.
2. Is the sequence $\left(\frac{3}{2}\right)^n \mod 1$ dense in the unit interval?
3. Is it true that
$$\sum_{n=0}^{\infty} \frac{1+14n+76n^2+168n^3}{2^{20n}} \binom{2n}{n}^7 = \frac{32}{\pi^3}$$
4. *(The following is called Giuga Conjecture or Agoh-Giuga Conjecture and its origins can be traced back in* 1950.*)* A positive integer $p > 1$ is prime if and only if
$$\sum_{i=1}^{p-1} i^{p-1} \equiv -1 \pmod p$$
5. Why is it so difficult to prove that $e+\pi$ is irrational?
6. Let $\left(\frac{n}{7}\right)$ denote the Legendre symbol. Is it true that
$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log\left|\frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}}\right| \, dt = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{1}{n^2}$$
7. Is the Catalan's constant defined as
$$\mathcal{G} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$
irrational?
8. Let $\mathcal{H}_n$ denote the n - th Harmonic number. Is it true that for all $n \geqslant 1$ it holds that
$$\sum_{d|n} d \leqslant \mathcal{H}_n + (\log \mathcal{H}_n) e^{\mathcal{H}_n}$$
9. Let $x_0 = 2$. Is it true that the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined as
$$x_{n+1} = x_n - \frac{1}{x_n}$$
is unbounded?
10. Does the series
$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$
converge?
11. Is it true that
$$\lim_{n\to+\infty} \frac{1}{n^2 \sin n} = 0$$
12. Let p_n denote the n -th prime. Is the series
$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{p_n}$$
convergent?
13. Is there a dense subset of a plane having only rational distances between its points?
14. For every odd prime is it true that one has
$$0! + 1! + \cdots + (p-1)! \not\equiv 0 \pmod p$$
15. *(The following is known as Littlewood's conjecture.)* For $\alpha\ ,\ \beta \in \mathbb{R}$ is it true that
$$\liminf_{n\to+\infty} (n \cdot ||n\alpha|| \cdot ||n\beta||) = 0$$
Here $||\cdot||$ denotes the distance to the nearest integer.
16. What is the largest possible volume of the convex hull of a space curve having unit length?

This kind of identity is amenable in principle to automatic theorem-proving methods, but (using known techniques) is out of reach of current computers. Another such formula is the Cullen's Pi Formula that can be found here.

Actually Jeff Lagarias showed that this is equivalent to the Riemann hypothesis!

We would expect this to tend to zero, but the proof is beyond what is currently known. It is expected that the irrationality measure of π is 2 (it is known that all but a zero-measure set of real numbers have irrationality measure 2). Therefore, it is expected that the sequence tends to 0 but currently there is no proof for that.

The origin of this problem traces back to Paul Erdős .

This is known as Kurepa's conjecture. A proof was claimed and published in 2004 but the claim was withdrawn in 2011.

References

Here is a list of references that indicate , potentially , the source of the majority of the problems or that of the appendix.

International Fora

- Mathematics Stack Exchange

  Description: Mathematics Stack Exchange is a Q&A site that allows users to ask and answer questions. It is quite rich in interesting questions of all levels from trivial up to very challenging ones.

- Art of Problem Solving

  Description: Art of Problem Solving (abbrev: AoPS) is a site that is a great resource of mathematical competitions. It also has a college forum with plenty of interesting questions and answers.

- mathimatikoi.org/forum

  Description: mathimatikoi.org (from the greek word that means mathematicians) is an English forum of university mathematics. Its main focus is in college level mathematics and some branches of Euclidean Geometry.

- Integrals and Series

  Description: Integrals and Series is a forum on discussion on Integrals and Series only. It has many topics on the evaluation of challenging integrals and series as well as studies on special functions.

  Note: *This site / forum is using Tapatalk and MathJaX is no longer rendering math equations. You are **strongly** adviced to use a bookmark so that it renders MathJaX. Unfortunately , this site (which once was a valuable resource of integrals and series) is useless anymore.*

Local Fora

- mathematica.gr

  Description: mathematica.gr is a greek site on mathematical discussions. It is a great resource on mathematical competitions , mathematical news, teaching technics as well as university and applied mathematics.

Other Sites

- tolaso.com.gr

  Description: The editor's personal site.

Institutions

- University of Ioannina, Ioannina, Greece
- University of Athens , Athens, Greece
- University of Wisconsin , USA
- University of Michigan, Michigan , USA

Books / Journals

- American Mathematical Monthly
- Romanian Mathematical Monthly
- Asymmetry
- Rudin W. Principals of Mathematical Analysis
- Principals of Multivariable Calculus , Giannoulis Ioannis , University of Ioannina
- Complex Analysis , Stein E.M and Shakarchi R

Other References

These other references may include facebook groups.